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RESOLUTION REFINEMENT TECHNIQUE IN A SMOOTHED PARTICLE HYDRODYNAMICS NUMERICAL FLUME FOR COASTAL ENGINEERING APPLICATIONS

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Abstract. Numerical modeling of the wave interaction with coastal structures is a challenging issue due to the multi-nonlinear phenomena involved, such as, wave propagation, wave transformation, interaction among incident and reflected waves, run-up / run-down, wave breaking and wave overtopping. Numerical models based on a Lagrangian formulation, like SPH (Smoothed Particle Hydrodynamics), allow simulating complex free surface flows. This work presents the new developments on a SPH numerical model for studies on wave-structure interaction made at the National Civil Engineering Laboratory (LNEC). A new semi-automatic refinement technique of particles was applied to reduce the CPU time. Simulations with a specific geometry, a wave flume with a water chamber, were made regarding the application of this technique. An analysis was made on (i) convergence with resolution, i.e. particle dimension and (ii) semi-automatic refinement. Results were compared with the solution obtained with the finer resolution.

1 INTRODUCTION

Some of the maritime structures intend to use the wave action effects for the benefit of the communities as an advantage for the local economy. The study regarding the response of these structures is therefore important to ensure both their stability and functionality. Wave-structure interaction generates very complex phenomena involving nonlinear processes, like wave propagation and transformation, run-up / run-down, wave breaking and overtopping. Coastal structures may have different structural characteristics. The analysis of the hydraulic behavior of the structures is usually made by using semi-empirical formulae. However, direct application of these formulae is limited to simple structural configurations and to specific wave conditions. In practical engineering studies, physical modeling tests are usually also undertaken, which permit a reliable evaluation of the structure's efficiency. The greatest disadvantages of the physical model tests are the required time for their construction and exploitation, the high cost and lack of flexibility to change the geometrical characteristics of the models. Moreover, scale effects could affect the results.

In recent years, due to the continuous increase in computational power, numerical models have been developed and their use has become increasingly attractive. Numerical modeling of the free-surface flow has attracted the interest of a large scientific community and wave interaction with coastal structures is a challenge due to the nonlinear phenomena involved.

The equations describing the flow are known for a long time, but with the improvement and the development of the computational techniques it has become easier to obtain approximate solutions to these equations and consequently to simulate realistic scenarios.

However, only a few numerical models allow simulating the very complex phenomena of wave breaking, overtopping and impact loads at vertical structures. Among the existing models, one can highlight the three different types of models that are currently in development and/or validated at LNEC: AMAZON [9], based on the nonlinear shallow water equations; IH-2VOF [2], based on the VARANS equations (Volume-Average-Reynolds-Navier-Stokes), and the SPHyCE [4] based on a Lagrangian method and the concept Smoothed Particle Hydrodynamics (SPH). The models AMAZON and IH-2VOF have already been successfully applied for wave-structure interaction studies, and the SPHyCE model is under development and validation.

Recently, models based on Lagrangian methods, such as the SPH approach, have emerged. This method is based on the Navier-Stokes equations and on a completely mesh-free technique. Monaghan [12] demonstrated that SPH is a very promising alternative for modelling free surface flows and wave breaking. Different models have been developed, based on the SPH formulation of the Navier-Stokes equations, and there are many different numerical implementations. The SPHyCE numerical model used and developed at LNEC is based on the original SPHysics model [8] and specially developed and improved for studies of wave interaction with impermeable and porous coastal structures and for applications of coastal engineering. Promising agreement with experimental data has been obtained for the free surface elevation, overtopping discharge and impact loads [2], [3], [4], [5] [6]. The present numerical model includes three specific developments, among others: i) a partial renormalization, i.e. partial filtering density, where renormalization is applied only for particles near the structure, which is an original method that allows simultaneously propagating waves, without diffusion, and modeling accurately the pressure field near the structure [5]; ii) an wavemaker with active absorption with paddle drift correction allowing the simulation of a semi-infinite numerical wave flume; and iii) a new semi-automatic refinement technique by the division of fluid particles. This work will be focused mainly on the semi-automatic refinement technique.

The analysis of the semi-refinement technique is fundamental to assure the limitations and potentialities of the new implemented technique in order to reduce the calculation time and to improve the model results.

To study the semi-automatic refinement technique, a simple water chamber was simulated. This geometry was chosen to avoid excessive nonlinear effects such as wave breaking, impact loads on a vertical wall and wave transformation over a sloping structure. The idea was to simulate wave propagation regarding the semi-automatic refinement study, considering both an over simplified and a high complex wave propagation, focusing mainly on the wave effects that are directly connected with the new implemented technique of the semi-automatic refinement. An analysis of this technique was done, a convergence study with the resolution, i.e. the particle dimension, was performed to define the compromise between results accuracy and CPU time, and an analysis of semi-automatic refinement technique is carried out for several resolutions.

Therefore in section 2 the fundamental principle of the SPH method is explained and the new improvements of the SPH model are described. In section 3 the case study is described. In section 4, the free surface in the flume and the water chamber are analysed using the semi-automatic refinement technique. The conclusions are presented in the last section.

2 NUMERICAL MODEL

2.1 SPHyCE numerical model

The SPH method [12] is based on a Lagrangian formulation of Navier-Stokes equations. It is a mesh-free technique which allows modeling fluid particle trajectories. Numerically, the interaction between the particles is ensured by an interpolation function, the kernel [11].

Lagrangian Navier-Stokes equations are transformed into SPH forms, by integral equations using integral interpolants, which allows approximating any function *A(r)* by:

$$A(r) = \int_{\Omega} A(r')W(r - r', h)dr' \tag{1}$$

where r is the vector particle position, W is the kernel (weighting function), h is the smoothing length. The kernel allows determining the interaction among neighbouring particles included in the influence domain, a compact support within a circular region determined by a radius of $2h$, controlled by the smoothing length h, typically higher than the initial particle spacing, d_0.

The two-dimensional SPH equations are based on the Lagrangian formulation of the conservation of momentum and continuity:

$$\frac{dv}{dt} = -\frac{1}{\rho}\nabla P + \Pi + g \tag{2}$$

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\mathrm{div}\ (v) \tag{3}$$

where t is the time, Π represents the viscous terms, g = (0, -9.81) ms^{-2} is the acceleration of gravity, v, P and ρ are the velocity, pressure and density, respectively.

The standard SPH formulation [12], in which the fluid is considered weakly- compressible, is used and allows calculating the pressure by an equation of state [1]:

$$P = B\left[\left(\frac{\rho}{\rho_0}\right)^{\gamma} - 1\right] \quad \text{with} \quad B = \frac{c_0^2 \rho_0}{\gamma} \tag{4}$$

with γ=7, ρ_0 the reference density (for water: 1000kg.m^{-3}) and c_0 the sound speed.

The trajectories of the particles are obtained from the following relationship:

$$\frac{dr}{dt} = v \tag{5}$$

SPHyCE numerical model, based on the SPHysics code [8], has been developed and improved for specifically solving coastal engineering problems and modeling complex free surface flows and wave interaction with coastal structures (impermeable and porous structures).

For numerical simulations of wave propagation and interaction with a vertical wall, the quadratic kernel is used to determine the interaction between the particles.

Integration in time is performed by the Predictor-Corrector model using a variable time step to ensure the CFL condition.

Particles are usually moved using the XSPH variant due to Monaghan [12]. The method is a correction for the particle velocity, which is recalculated taking into account the velocity of that particle and the average velocity of neighbouring particles. However, it was shown in [3] that instabilities appear during wave propagation due to the XSPH correction and fluid flow exhibits unphysical behaviours. Consequently, the XSPH correction is not used.

While the kinematics of SPH simulations is generally realistic, the pressure field of the particles can exhibit large pressure oscillations. One of the most straightforward and computationally least expensive methods to smooth out pressure oscillations is to perform a filter over the density of the particles, using the Shepard density filter, and to re-assign a density to each particle [5]. In the present study, SPHyCE model is applied using a partial renormalization technique developed in [5] and [6].

The boundary conditions are not displayed directly in the SPH formalism. In the present model the repulsive boundary condition, developed by [13], that imposes a repulsive force to the fluid particles from the boundary particles, is used and allows preventing the water particles to cross the solid boundary. Nevertheless, reinforcing this condition, some improvements were made in the SPHyCE model.

Initially, the water particles are placed in the flume using a regular Cartesian grid, i.e. particles are regularly distributed, with the spacing between particles defined by d_o. This is a condition of the SPH method when the smoothing length of the kernel is constant. The distribution of the solid particles at boundaries follows the one adopted for the fluid particles, namely the distance between the particles is equal to d_o independently of the boundaries direction. The velocity field is zero and the pressure is hydrostatic.

2.2 Semi-automatic refinement technique

A new semi-automatic refinement technique by division of fluid particles, in order to decrease the computational time without loss on results accuracy, was implemented in the SPHyCE code and analysis of this technique is the aim of the present work.

The model enables the division of one into 2, 3 or 4 smaller particles at any time during the calculations, allowing the refinement when it is most needed (Figure 1).

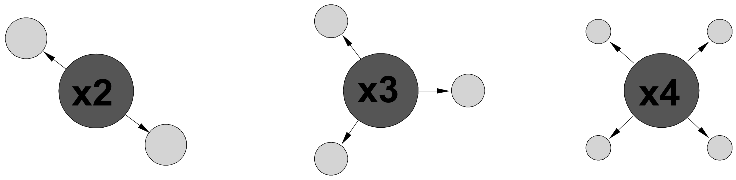

Figure 1: Division of the particles into 2, 3, or 4 smaller particles

This technique allows SPH simulations to start with a coarse resolution, producing relatively short CPU time in order to model the transient part of the flow, i.e. before stabilizing the interaction between the incident and reflected waves by the structure. After obtaining this flow stabilization the refinement technique is applied in order to split all the fluid particles in the computational domain (Figure 2), increasing the resolution in order to obtain more accurate results. This technique is to be applied for studies on a semi-infinite wave channel where the transient flow is not relevant to the results analysis.

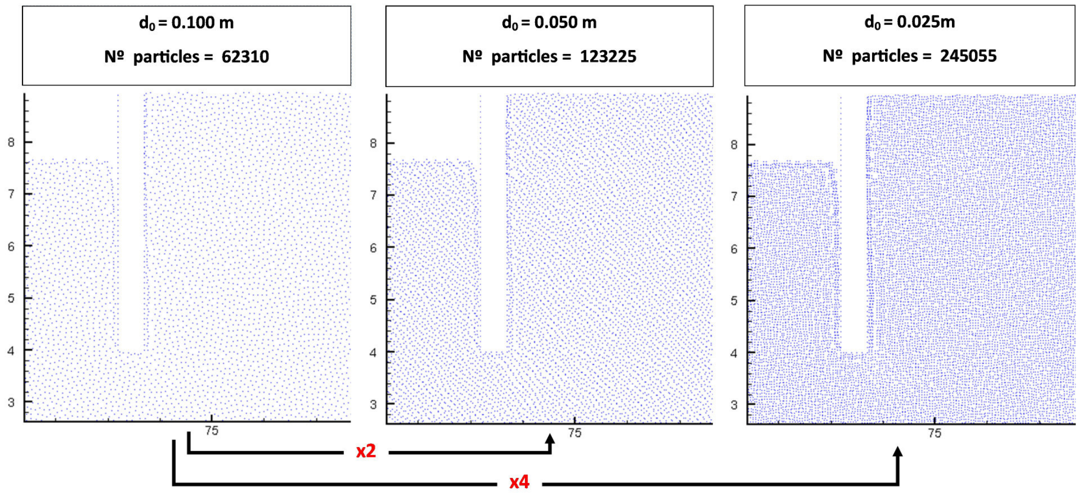

Figure 2: Division of the fluid particles into 2 and 4 smaller particles

3 CASE STUDY

A specific geometry for the computational domain was defined to study accurately the new implemented semi-automatic refinement technique.

The case study presents a wave flume with a water chamber to assess the influence of the new refinement technique for the wave propagation and for the nonlinear interaction with a simple structure.

The flume (Figure 3) is 80.0 m long and has a horizontal bottom. The water chamber begins at the distance of 72.3 m from the wavemaker and has a first vertical wall of 0.5 m thick and 5.0 m height and a second wall, coincident with the end of the flume. The first wall of the water chamber was defined to be submerged up to 4.0 m deep.

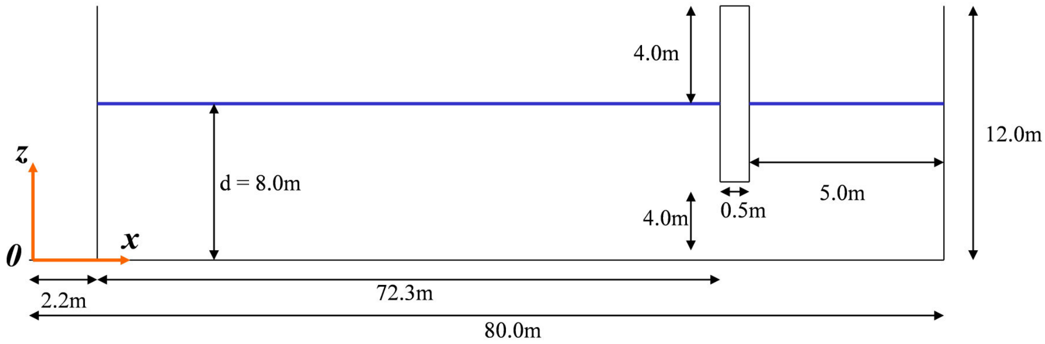

Figure 3: Schematic representation of the wave flume and water chamber

An incident regular wave was tested with 7.0 s wave period, *T*, and 2.0 m height wave, *H*. Water depth, *d*, is 8.0 m, which result in a wave length, *L*, equal to 55.2 m.

Active absorption of the reflected waves at the wavemaker was used in the numerical simulations and allows modeling a semi-infinite wave flume with a small computational domain, with approximately 1.3*L* length.

In order to study the free surface inside and outside the water chamber, 8 numerical wave gauges were placed along the wave flume (Figure 3). Two gauges, G1 closer to the wavemaker at *x*=8.2 m and G3 close to the front face of the wall at the water chamber entrance at *x*=72.9 m, and the mean level inside the water chamber are presented in this work.

The partial filtering density technique [6] [8] was applied to particles from *x*=70.0 m (before the wall that separates the water chamber) to *x*=80.0 m (inside the water chamber) each 30 time iterations.

4 RESULTS

As referred, in the present work the free surface at gauge G3 and the mean level inside the water chamber are presented and compared.

For the semi-automatic refinement study, two different analyses were made: (i) convergence study with particle dimensions, d_o, varying from 0.2 m to 0.05 m (i.e. 15918 to 246612 particles); and (ii) analysis of the semi-automatic refinement technique, comparing results with and without the semi-automatic refinement.

The mean free surface elevation inside the water chamber was calculated from the water volume inside the water chamber.

4.1 Convergence analysis

A convergence study is here performed varying the resolution, i.e. the particle dimension or/and particle volume. Table 1 shows the characteristics of the five computational domains used to check independence of the solution to the resolution. In this table, particle dimension, d_o, particle volume, total number of particles for each resolution and the number of particles per wave height are indicated ([5] specify that 40 particles are the minimum for a convergent solution in SPH simulations).

Figure 4 shows the time series of the free surface elevation outside the water chamber (G3) and the mean level inside the chamber.

Table 2 shows the average wave height at G3 and the mean level inside the water chamber for the five tested resolutions. CPU time, using a serial version of the SPHyCE code and a Personal Computer Intel(R) Core(TM) i7 CPU 920 @ 2.67GHz, and relative errors are also presented considering the solution obtained for the finer resolution (5A).

Table 1: Particle characteristics and number of particles used for the computational domain

Case	*do* (m)	Particle volume (m^3/m)	Number of particles	Number of particles per wave height
1A	0.200	4.00×10^{-2}	15918	10
2A	0.143	2.04×10^{-2}	30822	14
3A	0.100	1.00×10^{-2}	62310	20
4A	0.071	5.00×10^{-3}	123609	28
5A	0.050	2.50×10^{-3}	246612	40

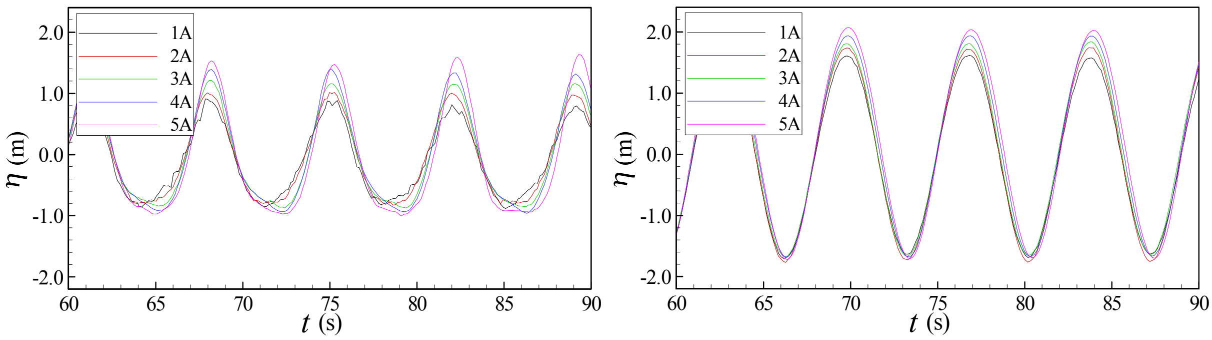

Figure 4: Convergence analysis with resolution - Free surface elevation at gauge G3 and mean level inside the water chamber

Table 2: Convergence analysis with resolution – CPU time, average wave height at G3 and mean wave heigth in the water chamber and relative errors

		Free surface at gauge G3		**Mean free surface in water chamber**	
Case	CPU time (hours)	Average wave height (m)	Relative error (%)	Average level (m)	Relative error (%)
1A	5.0	1.690	31.313	3.264	13.026
2A	15.0	1.783	27.539	3.469	7.558
3A	45.0	2.027	17.587	3.473	7.464
4A	131.3	2.294	6.746	3.588	4.382
5A	370.0	2.460	0.000	3.753	0.000

The convergence analysis demonstrates that, as expected, the quality of the results increases with the resolution. The results for the free surface elevation at gauge G3 reveal that the relative error can be reduced from 31% (1A) to 7% (4A) and for the mean level in the water chamber from 13% (1A) to 4% (4A), when comparing with the more refined case.

Differences are small between case 4A and 5A, particularly inside the water chamber. The CPU time increases sharply with the number of particles in the domain, as can be seen in Table 2. From case 1A to 5A the calculation time growths about 74 times, from 5 hours to 370 hours.

Figure 4 shows the CPU time versus the number of particles. Increasing the number of particles by a factor 2 induces that CPU time is multiplied by a factor 3. Based on the five resolutions, a trend line indicates that CPU time growths following the relationship $3x10^{-6} N^{1.55}$, with N the number of particles.

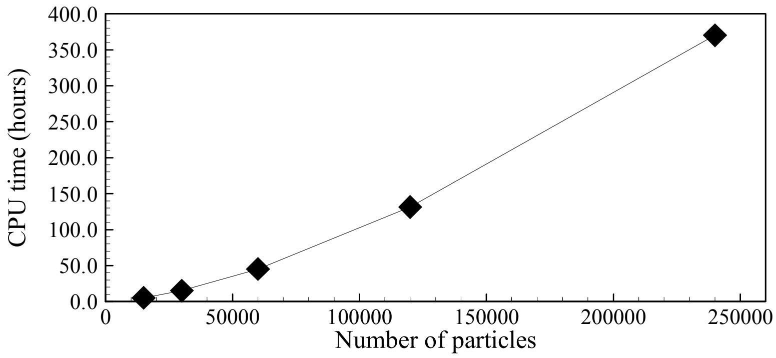

Figure 5: CPU time versus number of particles for the five resolutions

4.2 Semi-automatic Refinement

For the semi-automatic refinement technique, four refinements were obtained from two different particle resolutions, the 2A and 3A respectively, and both resolutions were refined by 2x and 4x. In Table 3 there are 4 cases to assess the semi-automatic refinement performance: (i) 2A-2x case is comparable with the 3A; 2A-4x case is analogous to the 4A; 3A-2x case is similar to the 4A; and finally the 3A-4x case is equivalent to the 5A case. Both resolutions will be compared with the respective simulations without refinement.

The refinement technique was applied after 55 s of simulation, when stabilized wave regime was obtained, considering a total simulation of 105 s.

Table 3: Refinement analysis - Cases with and without the semi-automatic refinement

Case	*do* (m)	Number of particles	Number of particles per wave height
2A	0.143	30822	14
2A-2x	0.101	60666	20
2A-4x	0.071	120354	28
3A	0.100	62310	20
3A-2x	0.071	123225	28
3A-4x	0.050	245055	40
4A	0.071	123609	28
5A	0.050	246612	40

In order to compare the non-refined with the refined test cases, Figure 6 to Figure 8 present the free surface elevation outside the water chamber (G3) and the mean level inside the water chamber for the eight simulations.

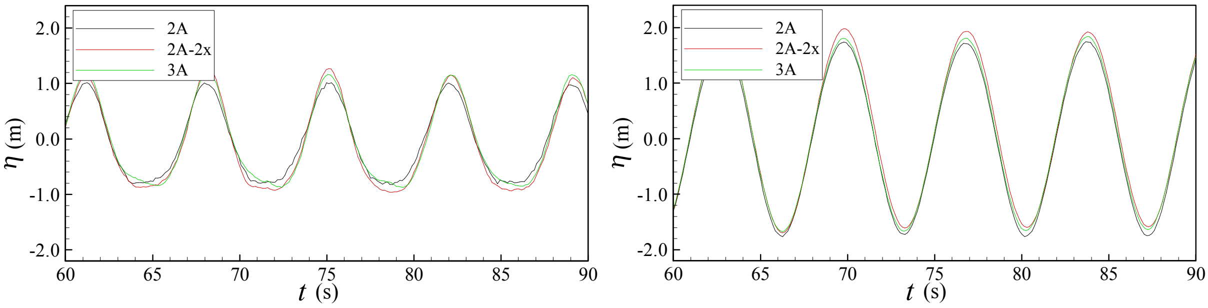

Figure 6: Refinement analysis **-** Free surface elevation at gauge G3 and mean level inside the water chamber for cases 2A, 2A-2x, and 3A

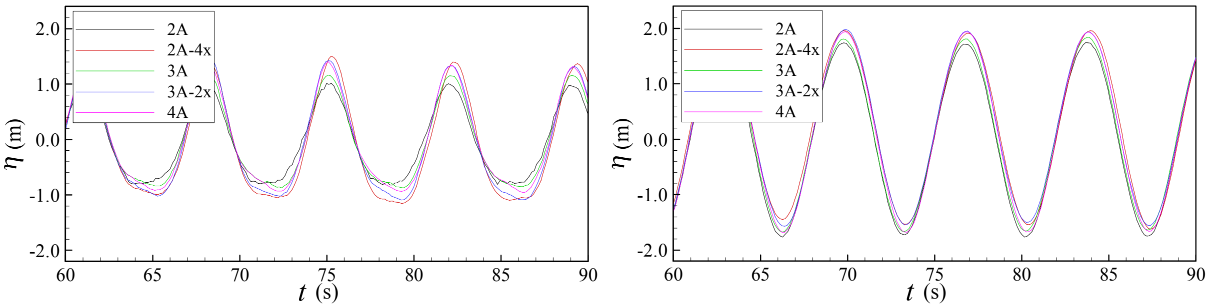

Figure 7: Refinement analysis **-** Free surface elevation at gauge G3 and mean level inside the water chamber for cases 2A, 2A-4x, 3A, 3A-2x and 4A

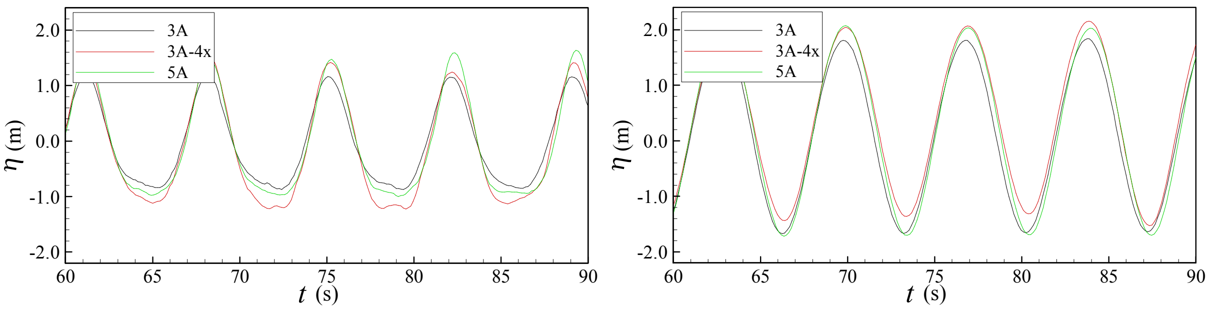

Figure 8: Refinement analysis **-** Free surface elevation at gauge G3 and mean level inside the water chamber for cases 3A, 3A-4x, and 5A

Table 4 shows the CPU time, the average wave height at G3 and the average height of the mean level in the water chamber for the 8 cases presented in Table 3. These 8 cases were grouped into 3 comparing tables to better analyze the effect of the semi-automatic refinement technique. Relative errors are also presented comparing the refined and non-refined cases with those with the highest initial resolutions.

The objective of the semi-automatic refinement technique is to firstly use a less refined calculation with the purpose of having a quickly stabilized wave regime and then, using the semi-automatic refinement, simulate a well refined case in order to have approximately the same solution as the obtained with a more refined simulation (that would take a greater calculation time if using the same refinement for the whole simulation). Hereby, the new technique leads to a decreasing on the calculation time and maintains the same quality of the results.

The results presented in Table 4 show an agreement with the above hypothesis. In fact, observing the results, in most of the cases, the use of the semi-automatic refinement technique produces a solution that approximates to an even more refined simulation.

Table 4: Refinement analysis – CPU time, average wave height at G3 and mean wave height in the water chamber and relative errors

				Free surface at gauge G3		Mean free surface in water chamber	
	Case	Number of particles	CPU Time (hours)	Average wave height (m)	Relative error (%)	Average height for the mean level(m^3)	Relative error (%)
Evaluation 1	2A	30822	15.0	1.783	12.08	3.470	6.29
	2A-2x	60666	27.5	2.101	3.63	3.522	7.90
	3A	62310	45.0	2.027	0.00	3.264	0.00
Evaluation 2	2A	30822	15.0	1.783	22.30	3.470	3.32
	2A-4x	120354	68.8	2.423	5.63	3.538	1.41
	3A	62310	45.0	2.027	11.63	3.264	9.04
	3A-2x	123225	85.0	2.423	5.61	3.491	2.74
	4A	123609	131.3	2.294	0.00	3.589	0.00
Evaluation 3	3A	62310	45.0	2.027	17.59	3.264	13.03
	3A-4x	245055	215.6	2.571	4.51	3.616	3.65
	5A	246612	370.0	2.460	0.00	3.753	0.00

Considering Evaluation 1 at gauge G3, by doubling the number of particles (2A-ex) the solution reduces the relative error from the most refined solution (3A) from 12.08% to 3.63%. Inside the water chamber, the semi-automatic refinement produces the opposite effect, increasing the error from 6.29% to 7.90%. This might be due to the fact that these cases present a low resolution, with maximum of 20 particles per wave height that is considerably lower than the desirable for an acceptable solution [5].

For Evaluation 2, resolution is about 28 particles per wave height and results are more accurate. For the gauge G3, quadruplicating the number of particles of the case 2A (2A-4x), produces an increase of the results accuracy, reducing the relative error from 22.30% to 5.63%. The same tendency is showed by doubling the number of particles of the solution 3A (3A-2x) producing a decrease of relative error from 11.63% to 5.61%.

Finally Evaluation 3 presents simulations with 40 particles per wave height providing a suitable solution for the tested case [5]. Both gauge G3 and mean level inside the water chamber present better results when quadruplicating the number of particles of the 3A (3A-4x). At G3 and inside the water chamber, the relative error decreases from 17.59% to 4.51%

and from 13.03% to 3.65%, respectively.

Regarding the calculation time, main focus of the semi-automatic refinement technique, Table 4 proves a real improvement on the calculation time, with a CPU time reduction of about 35% to 47% when refining the particles by a factor of two or four. Comparing the case 3A-4x and 5A there is less than 5% differences between results and CPU time is 42% reduced, decreasing from 370 to 215.6 hours.

5 CONCLUSIONS

The paper presents the study and development of a new semi-automatic refinement technique based on the division of fluid particles. The case study presents a simplified structure of a wave flume with a water chamber. This structure allows analyzing of the wave propagation and its specific interaction with a vertical wall and a water chamber.

The analysis of free surface elevation near the entrance of the water chamber and mean level inside the water chamber are presented for: (i) the convergence with particles resolution; and (ii), the main purpose of this work, the semi-automatic refinement technique regarding two different resolutions and two types of refinement, doubling and quadrupling the number of particles.

The convergence analysis shows a clear convergence of the solution with the refinement of particles; the two most refined simulations (4A and 5A) only present a relative difference of 7% and 4% for the gauge G3 and the mean level inside the water chamber, respectively. However, for these two refined simulations, CPU time increases from 131.3 to 370 hours, i.e. doubling the number of particles of the CPU time reduces around 3 times.

Three different evaluations were made for the analysis of efficiency of the new semi-automatic refinement technique, considering a final resolution of about 60000, 120000 and 240000 particles, and multiplying by two and four the number of particles of the two coarser resolutions. The results prove that the CPU time using the refinement technique decreases considerably maintaining good agreement of the results when compared with those obtained without refinement. All simulations with the semi-automatic refinement present relative differences with the solution obtained with the finer resolution less than 6% for the free surface elevation outside and level inside the water chamber. Refining the particles by a factor of two or four the CPU time is considerably reduced, by 35% and 47%, respectively.

The semi-automatic refinement results show that the new implemented technique improves significantly both the final solution and the computational time.

For the future work, regarding the semi-automatic refinement technique further analyses are needed, other geometries and other physical phenomena (as wave overtopping) will be studied. Also the comparison and validation with physical tests will be performed.

6 ACKNOWLEGEMENT

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