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EVALUATION OF MEASUREMENT UNCERTAINTY IN TESTS REQUIRING NON-LINEAR REGRESSION

J. A. Sousa[1], I. M. Smith[2], A. B. Forbes[2], A. S. Ribeiro[3], D. F. Sousa[1]

[1]Laboratório Regional de Engenharia Civil, IP-RAM, 9000-264, Funchal, Portugal
[2]National Physical Laboratory, Teddington, TW11 0LW, United Kingdom
[3]Laboratório Nacional de Engenharia Civil, 1700-066, Lisboa, Portugal

Abstract: This work is concerned with the evaluation of measurement uncertainty arising from the use of non-linear regression in testing. While modern standards require a measured value to be accompanied by a statement of its associated uncertainty, for many applications existing work procedures make no reference to uncertainty. Such procedures should therefore be updated to take account of the requirement to provide uncertainty information. This paper considers an application that involves the use of non-linear regression and for which uncertainty evaluation does not constitute part of the current work procedure. An updated procedure is proposed and example results are presented.

Keywords: Uncertainty, non-linear regression, modelling, least squares adjustment.

1. INTRODUCTION

Nowadays, according to the international standards regulating work in laboratories [1] and the presentation of results in testing [2][3], virtually all fields of science require that a measured value be accompanied by an associated uncertainty. For many applications, however, the standards that specify work procedures were written prior to the publication of the "Guide to the expression of uncertainty in measurement" (GUM) [4] and make no reference to the evaluation or presentation of uncertainties. Despite the modern requirement to provide uncertainty information, many engineering testing laboratories still do not provide such information as part of their measurement results.

This paper considers an application, concerned with soil compaction, for which the work procedure [5] does not describe uncertainty evaluation, and discusses how the procedure may be adapted to account for measurement uncertainty. The application requires a function to be fitted to measured data followed by the determination of quantities dependent on the parameters of the fitted function.

For the application, background is provided together with details of the mathematical problem to be solved using the approach described in the work procedure. A proposal is made regarding the updating of the procedure to take account of uncertainty associated with the measured data and to evaluate uncertainties associated with the outputs of the procedure. Example results are presented to illustrate the implementation of the updated procedure.

2. SOIL COMPACTION TESTS

Compaction of soil is a mechanical process, consisting of fast and repeated application of a vertical load. The solid particles become more closely packed together, thus increasing the dry density of the soil and leading to a larger area of contact among the solid particles and an increased capacity to withstand loads. The dry density that can be achieved depends on the amount of moisture present in the soil and, for a given degree of compaction, there is an optimum moisture content at which the dry density reaches a maximum value.

Compaction tests are of great practical use, namely in the determination of reference parameters (e.g., optimum moisture content and maximum dry density) normally used in the control of results obtained during compaction works *in situ*.

Testing involves the compaction, in layers, of a soil sample within a compaction mould of specified dimensions, by means of an applied number of blows from a rammer of normalized weight, dropping from a normalized height, over the surface of each of the soil layers in the mould. This procedure is repeated for different amounts of moisture present in the soil in order to obtain six specimens.

According to the work procedure in the standard concerned with compaction-related tests [5], for each compacted specimen, the measured values of dry density (denoted by γ_s) are plotted against the corresponding measured values of moisture content (W). A curve of best fit (compaction curve) to the six points is drawn and, using this curve, estimates of the maximum dry density and the corresponding optimum moisture content are determined. Clearly the nature of the best-fit function will influence the estimates of maximum dry density and optimum moisture content of the soil. However, the procedure provides no guidance as to what type of function to use.

For this study, duplicate tests were carried out by three different laboratory technicians giving a total of six tests, the results for two of which are shown in Table 1. Figure 1 illustrates the compaction curves for each of the six tests obtained by fitting a degree three polynomial (cubic) function to the measured data.

Table 1 – Example measurement data

Specimen	Technician 1		Technician 3	
	Test 1		Test 6	
	W (%)	γ_s (g/cm^3)	W (%)	γ_s (g/cm^3)
1	24,4	1,20	24,7	1,21
2	26,8	1,28	26,9	1,30
3	28,0	1,35	29,6	1,37
4	29,0	1,37	31,8	1,38
5	30,6	1,38	33,8	1,33
6	33,0	1,31	35,1	1,30

Table 2 shows, for each test, the optimum moisture content value $W_{(opt)}$ based on the respective compaction curve and the corresponding maximum dry density value $\gamma_{s(max)}$.

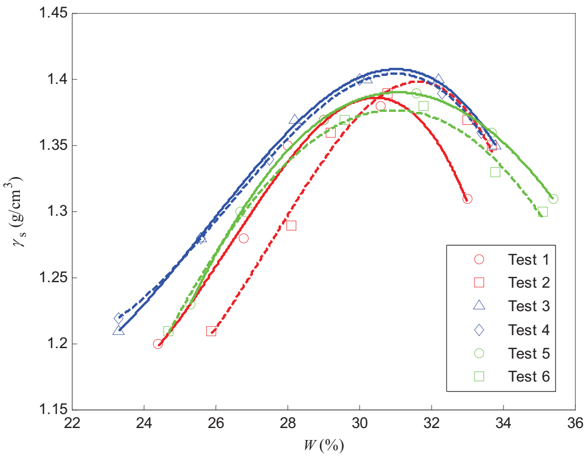

Figure 1 – Compaction curves for six tests

Table 2 – Reference parameters for six tests

Test	1	2	3	4	5	6
$W_{(opt)}$ (%)	30,5	31,6	31,0	31,0	31,1	31,0
$\gamma_{s(max)}$ (g/cm^3)	1,39	1,40	1,41	1,40	1,39	1,38

As mentioned in Section 1, the work procedure does not describe how uncertainties associated with the measured values could be used to determine uncertainties associated with the estimates of the reference parameters. A proposed approach that accounts for uncertainties is outlined below.

The moisture content value W_i for the ith specimen, expressed as a percentage, is given by the average of the moisture content values determined for two samples of the specimen. That is,

$$W_i = \frac{W_{i,1} + W_{i,2}}{2},$$

where

$$W_{i,j} = \frac{100\left(m_{2,i,j} - m_{3,i,j}\right)}{m_{3,i,j} - m_{1,i,j}},$$

where $m_{1,i,j}$ is the mass of the baseplate, $m_{2,i,j}$ is the combined mass of the baseplate and the jth bulk compacted sample of the ith specimen and $m_{3,i,j}$ is the combined mass of the baseplate and the jth dry compacted sample of the ith specimen.

The dry density $\gamma_{s,i}$ for the ith specimen is given by

$$\gamma_{s,i} = \frac{100\left(P_{T,i} - P_{M,i}\right)}{V_i\left(W_i + 100\right)},$$

where $P_{M,i}$ is the mass of the compaction mould, $P_{T,i}$ is the combined mass of the compaction mould and the ith bulk compacted specimen and V_i is the volume of the ith bulk compacted specimen.

The moisture content W_i may therefore be expressed as a function of the parameters $m_{1,i,1}$, $m_{1,i,2}$, $m_{2,i,1}$, $m_{2,i,2}$, $m_{3,i,1}$ and $m_{3,i,2}$, while the dry density $\gamma_{s,i}$ is a function of parameters $m_{1,i,1}$, $m_{1,i,2}$, $m_{2,i,1}$, $m_{2,i,2}$, $m_{3,i,1}$, $m_{3,i,2}$, $P_{M,i}$, $P_{T,i}$ and V_i. Given values of the parameters, together with uncertainties associated with those values, corresponding values of W_i and $\gamma_{s,i}$ and their associated uncertainties can be determined. The dependence of both W_i and $\gamma_{s,i}$ on parameters $m_{1,i,1}$, $m_{1,i,2}$, $m_{2,i,1}$, $m_{2,i,2}$, $m_{3,i,1}$ and $m_{3,i,2}$ means that they are correlated and a covariance term may be calculated that quantifies the effect of this correlation. In addition, the same compaction mould is used throughout each test, causing the quantities W_i and $\gamma_{s,i}$ for all six specimens to be correlated with each other. Further correlation arises from repeated use of balances during the test. Uncertainties and covariances associated with the values of W_i and $\gamma_{s,i}$ can be calculated and collected together in the covariance matrix V of dimension 12 × 12.

The compaction curve models the dry density γ_s as a function of the moisture content W and a set of parameters $\mathbf{b}$ that are dependent on the choice of model:

$$\gamma_s = f(W, \mathbf{b}).$$

The approach proposed in this paper takes into account the uncertainty and covariance information stored in the covariance matrix V and requires solving a *generalized Gauss-Markov regression problem* [6] to determine the compaction curve. Estimates $\hat{\mathbf{b}}$ of the model parameters and $\hat{W}_i$ of the moisture content values are calculated such that the generalized sum of squares of residuals

$$\begin{bmatrix} W_1 - \hat{W}_1 \\ \vdots \\ W_6 - \hat{W}_6 \\ \gamma_{s,1} - f\left(\hat{W}_1, \hat{\mathbf{b}}\right) \\ \vdots \\ \gamma_{s,6} - f\left(\hat{W}_6, \hat{\mathbf{b}}\right) \end{bmatrix}^{\mathrm{T}} V^{-1} \begin{bmatrix} W_1 - \hat{W}_1 \\ \vdots \\ W_6 - \hat{W}_6 \\ \gamma_{s,1} - f\left(\hat{W}_1, \hat{\mathbf{b}}\right) \\ \vdots \\ \gamma_{s,6} - f\left(\hat{W}_6, \hat{\mathbf{b}}\right) \end{bmatrix},$$

is minimized. Solution of this problem requires the use of an iterative algorithm.

Testing the consistency of the measured data with the fitted model can provide confidence that the choice of model is appropriate.

Uncertainty information associated with the estimates of the model parameters in the form of a covariance matrix $V_{\mathbf{b}}$ can be evaluated in terms of the covariance matrix V associated with the measured data.

Having determined estimates $\hat{\mathbf{b}}$ of the model parameters and associated covariance matrix $V_{\mathbf{b}}$, it is required to determine estimates of the reference parameters, i.e., the optimum moisture content and the maximum dry density. Depending on the choice of model used for the compaction curve, it may be necessary to implement an iterative procedure to obtain an estimate of the optimum moisture content. The corresponding estimate of maximum dry density is obtained straightforwardly by evaluating the model function at that moisture content value. This paper considers the case where a cubic function is used. The estimate of the optimum moisture content is obtained by solving a quadratic equation.

Two approaches for evaluating the uncertainty associated with the estimates of the reference parameters are discussed. The first approach is the GUM uncertainty framework [4] in which the models expressing the reference parameters in terms of the compaction curve parameters are linearized. The uncertainties and covariances associated with the compaction curve parameter estimates are then propagated through the linearized models to determine uncertainties associated with the estimates of the reference parameters. The second approach is a Monte Carlo method as described in Supplement 1 to the GUM [7] and implements the propagation of distributions through the models for the reference parameters, making no linear approximation as in the GUM uncertainty framework. Given the non-linearity of the models for the reference parameters, it is important that the results obtained using the two approaches be compared so that a decision may be made as to which approach to use in practice.

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