JOINT STATISTICS

Nuno Feodor Grossmann, Laboratório Nacional de Engenharia Civil (LNEC)

ABSTRACT

The paper aims at presenting some achievements of joint statistics. The different statistical distributions used for the description of the main parameters of the joint sets (attitude, intensity, area, and aperture) are introduced. For the attitude, a comparison of the existing distribution functions is made. For the intensity, the formulas for its determination are given, the applicable statistical models are presented, and the Poisson model for the occurrence of the discontinuities in rocks is discussed in depth. The statistical models for the distribution of the volume of the blocks of a rock mass are introduced. For the area, again, the formulas for its determination are given, and the applicable statistical models presented. Finally, the distribution function for the aperture is referred to, as well as a note on the roughness and waviness.

1. INTRODUCTION

Rock masses always present numerous discontinuity surfaces, which may be of genetic origin (contacts between crystals, schistosity planes, bedding planes, etc.), tectonic origin (microfissures, joints, faults, etc.), or other origins (fractures due to the daily thermal wave, etc.).
From the Rock Mechanics point of view, the discontinuity surfaces may be sorted into 2 groups:
- one, that assembles those discontinuity surfaces which, due to their great number, small dimension, or little variation of the mechanical properties (at the scale of the considered problem), although having an influence on the rock mass properties, do not prevent the use of the Mechanics of the Continuous Media for the study of the rock mass; and
- another one, that assembles the remaining discontinuity surfaces, i.e., those which call for the use of the Mechanics of the Discontinuous Media for the study of the rock mass.
The whole lot of the discontinuity surfaces belonging to the second group, are called the rock mass **jointing** and, usually, include the joints, the faults, the fractures along schistosity planes, the bedding planes, etc.
The large majority of the discontinuity surfaces occurring in a rock mass, are approximately plane, and, therefore, the orientation in the space of each one of them, called the **attitude**, can be defined by two parameters, habitually, the **strike** (σ) and the **dip** (δ) (Grossmann 1977).
In general, a rock mass presents discontinuity surfaces with all attitudes, but, as a rule, the great majority of those surfaces may be included in a relatively small number of **discontinuity sets**, which are characterized by the fact that all discontinuity surfaces of the same set have adjoining attitudes, i.e., all discontinuity surfaces of a set are approximately parallel.
The jointing of a rock mass is, therefore, usually characterized by the presence of a small number of discontinuity sets, and, additionally, a few discontinuity surfaces with a random attitude.
However, in most studies of the geometrical characteristics of the rock mass jointing, the problem is simplified, by reducing it to the determination of the occurring discontinuity sets, and the description of their geometrical parameters.
The geometrical parameters which, generally, are considered in the description of a discontinuity set, are the attitude, the intensity, the area, and the opening, these parameters being chosen due to the fact that the discontinuity surfaces are often modelled as prisms with a very small height in relation to the dimension of their bases.

2. ATTITUDE OF THE DISCONTINUITY SETS

2.1. Concept

The parameter attitude describes the orientation in the space of the discontinuity surfaces of the set, independently from their location, and assuming they are plane.
As has been said, the attitude of a discontinuity surface is, in general, quantified through the 2 parameters strike and dip.

2.2. Statistical Distribution

2.2.1. Isotropic models

Although graphical representations of the distribution of the jointing surface attitudes have been used in geotechnical studies for nearly 80 years (Müller, 1933), the first mathematical models for the distribution function of the discontinuity surface attitudes of a set date from only about 40 years (Watson, 1966).
These models were, however, all isotropic, i.e., for them, the probability density function [f(ω, ε)] is given by

$$f(\omega,\varepsilon) = A\,e^{B\,g(\varepsilon)}$$

as a function of the longitude (ω) and the colatitude (ε) of the discontinuity surface attitude (in a system of spherical co-ordinates, whose revolution axis is normal to the mean attitude of the considered discontinuity set); the 2 constants (A) and (B); and a function [g(ε)] of the colatitude (ε).
The probability density functions of the most important isotropic models are:
a) for the **Arnold distribution** (Arnold,1941)

$$f(\omega,\varepsilon) = \frac{k_A}{2\pi(e^{k_A}-1)}e^{k_A\cos\varepsilon}$$

with the parameter (k_A) which measures the dispersion. The Arnold distribution is the hemispheric counterpart of the erroneously often used Fisher or spherical normal distribution.
b) for the **Bingham distribution** (Bingham 1964)

$$f(\omega,\varepsilon) = \frac{1}{2\pi M\left(\frac{1}{2};\frac{3}{2};k_B\right)}e^{k_B\cos^2\varepsilon}$$

with the parameter (k_B) which measures the dispersion, and Kummer’s confluent hypergeometric function [M (1/2; 3/2; k_B)], with the parameters (1/2) and (3/2), and the variable (k_B).
c) for the **isotropic bivariate normal distribution** (Grossmann 1985)

$$f(\omega,\varepsilon) = \frac{1}{2\pi\sigma^2\left(1-e^{-\frac{\pi^2}{8\sigma^2}}\right)}e^{-\frac{\varepsilon^2}{2\sigma^2}}$$

with the standard variation (σ). For practical purposes, the above expression reduces to

$$f(\omega,\varepsilon) = \frac{1}{2\pi\sigma^2}e^{-\frac{\varepsilon^2}{2\sigma^2}}$$

In spite of the limitation imposed by the isotropy, the use of these models still finds support in the literature. This standpoint, however, is not justifiable, because the experience has shown that the large majority of the discontinuity sets occurring in the rock masses, presents an anisotropic distribution of the discontinuity surface attitudes, and several anisotropic models for that distribution have already been presented in the literature.

2.2.2. Anisotropic models

The anisotropic models for the distribution function of the discontinuity surface attitudes of a set can be sorted into 2 groups:
- one, that assembles the models which use the 2 habitual parameters of the attitude, the strike (σ) and the dip (δ), as variables of a plane bivariate normal distribution; and
- another one, that assembles the models which use as variables the longitude and the colatitude (in a system of spherical co-ordinates whose revolution axis is normal to the mean attitude of the considered discontinuity set).

The models of the first group, which still have supporters, lead to unsatisfactory results, especially when the discontinuity surfaces of the set are nearly horizontal. This is due to the fact that the used distribution assumes that the elementary area (dσ dδ) has a constant value, although, in reality, it is the elementary area (sinδ dσ dδ) which has a constant value on the spherical surface.

As concerns the models of the second group, only 2 are mentioned in the literature, the Bingham distribution (Shanley & Mahtab 1975) and the bivariate normal distribution on the tangent plane at the mean attitude (LNEC 1973b).

2.2.3. Bingham distribution

The probability density function [f(ω, ε)] of this mathematical model is given by (Bingham 1964)

$$f(\omega,\varepsilon) = \frac{1}{2\pi M\left(\frac{1}{2};\frac{3}{2};z\right)} e^{-(\zeta_1 \cos^2\omega + \zeta_2 \sin^2\omega)\sin^2\varepsilon}$$

as a function of the longitude (ω) (measured from the orientation with the minimum dispersion) and the colatitude (ε) of the discontinuity surface attitude (in a system of spherical co-ordinates, whose revolution axis is normal to the mean attitude of the considered discontinuity set); the 2 parameters (ζ_1) and (ζ_2), which measure the dispersion, and obey the relation

$$\zeta_1 \geq \zeta_2$$

and Kummer’s confluent hypergeometric function [M (1/2; 3/2; z)], with the parameters (1/2) and (3/2), and the variable (z), which is given by the matrix

$$z = \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix}$$

As is shown in Fig. 1, the Bingham distribution does not reject the existence, in any given discontinuity set, of discontinuity surfaces whose attitude is normal to the mean attitude of that set.

Due to this inconvenient, which the Bingham distribution shares with the major part of the mathematical models for the distribution function of the discontinuity surface attitudes of a set, the bivariate normal distribution on the tangent plane at the mean attitude began to be used at the LNEC.

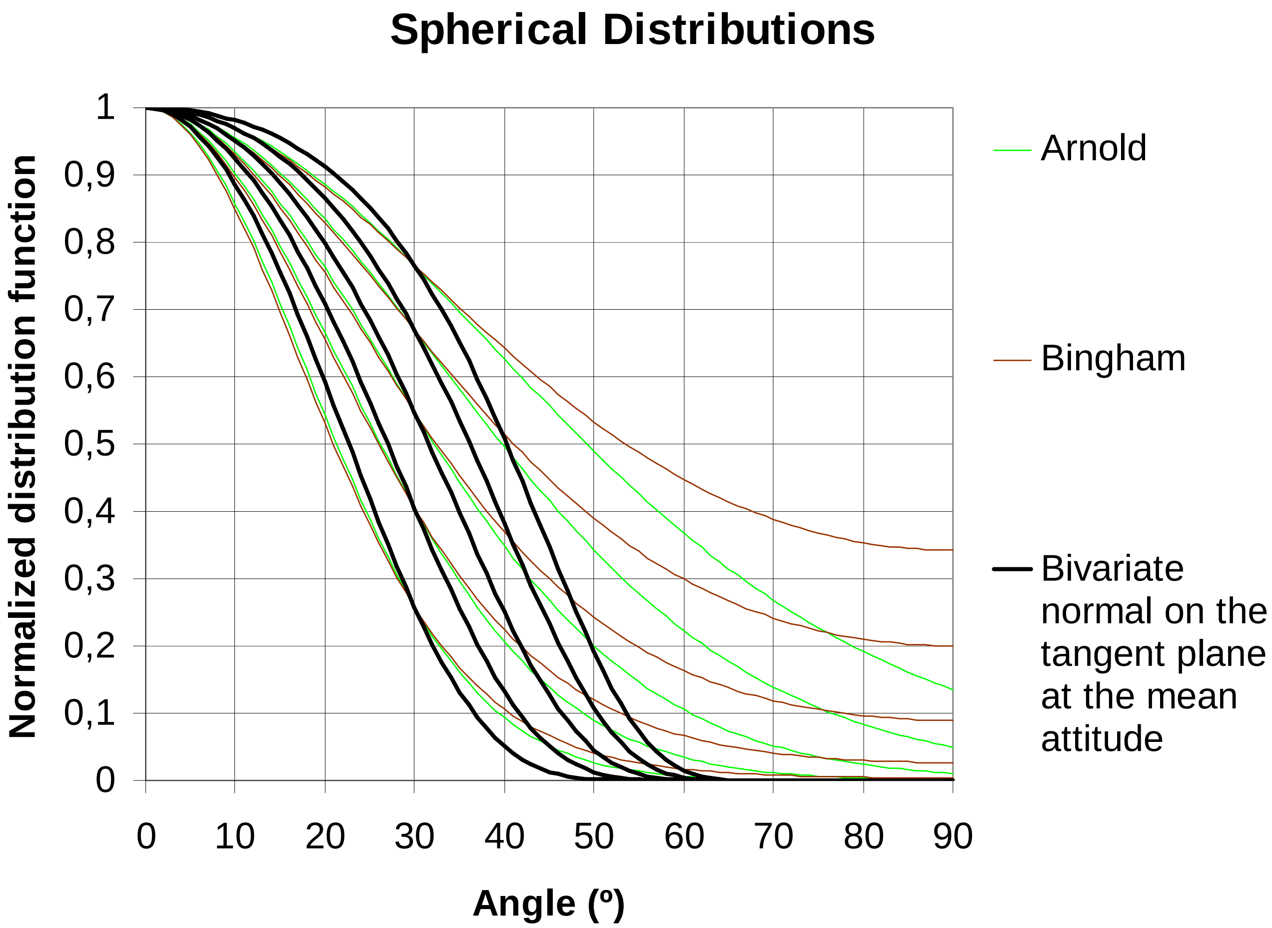

Fig. 1 – Normalized distribution functions of 3 spherical distributions

2.2.4. Bivariate normal distribution on the tangent plane at the mean attitude

On the spherical surface, the **probability density function** [f(ω, ε)] of this mathematical model is given by (Grossmann 1985)

$$f(\omega,\varepsilon) = \frac{1}{2\pi\sigma_M\sigma_m\cos^3\varepsilon} e^{-\frac{1}{2}\left[\frac{\cos^2(\omega-\omega_M)}{\sigma_M^2}+\frac{\sin^2(\omega-\omega_M)}{\sigma_m^2}\right]\tan^2\varepsilon}$$

as a function of the longitude (ω) and the colatitude (ε) of the discontinuity surface attitude (in a system of spherical co-ordinates, whose revolution axis is normal to the mean attitude of the considered discontinuity set); and the maximum standard deviation (σ_M), the minimum standard deviation (σ_m), and the longitude (ω_M) of the orientation with maximum dispersion, of the attitude distribution.

On the tangent plane at the mean attitude, the probability density function [$f_P(\omega, \varepsilon)$] of that distribution is given by (Grossmann 1977)

$$f_P(\omega,\varepsilon) = \frac{1}{2\pi\sigma_M\sigma_m} e^{-\frac{1}{2}\left[\frac{\cos^2(\omega-\omega_M)}{\sigma_M^2}+\frac{\sin^2(\omega-\omega_M)}{\sigma_m^2}\right]\tan^2\varepsilon}$$

and, thus, the area on that plane, limited by a line of equal probability density, for which the probability of having poles of the discontinuity surfaces of the considered set inside that area, is equal to (P), is the ellipse given by

$$\left[\frac{\cos^2(\omega-\omega_M)}{\sigma_M^2}+\frac{\sin^2(\omega-\omega_M)}{\sigma_m^2}\right]\tan^2\varepsilon = \ln\frac{1}{(1-P)^2}$$

In order to completely characterize the distribution of the discontinuity surface attitudes of a given set with the help of the bivariate normal model on the tangent plane at the mean attitude, one has, thus, to know **5 parameters**, the strike (σ) and the dip (δ) of the mean attitude, the maximum and the minimum standard deviations, respectively, (σ_M) and (σ_m), and the angle (ω_M) that identifies the orientation for which the maximum dispersion occurs.
The bivariate normal distribution on the tangent plane at the mean attitude is a unimodal distribution, which is symmetric in relation to the 2 perpendicular planes that correspond to the orientations for which the maximum and minimum dispersions occur. Its probability density function presents a **bell type shape**.
The 5 parameters defining the bivariate normal distribution on the tangent plane at the mean attitude, allow an easy visualization of the distribution, because:
- the mean attitude indicates directly the central point, which is the mode of the distribution;
- the maximum and minimum standard distributions give the limits between which lies the tangent of the colatitude of the points of the line of equal probability density which encloses the domain of attitudes containing

$$1-e^{-\frac{1}{2}}=39{,}3\%$$

of the poles of the discontinuity surfaces of the considered set; and
- the angle ω_M reveals directly the orientation for which the maximum dispersion occurs, and, due to the perpendicularity of their orientations, indirectly also the orientation for which the minimum dispersion occurs.

3. INTENSITY OF THE DISCONTINUITY SETS

3.1. Concepts

3.1.1. Intensity

The parameter intensity describes the degree of jointing that the whole lot of the discontinuity surfaces of the set have induce in the rock mass, independently of the individual extent of each discontinuity surface.
The intensity of a discontinuity set is, therefore, quantified by the sum of the areas of the discontinuity surfaces of the set which occur in a unit volume of the rock mass, and, so, should be expressed in m^2/m^3.
However, the intensity is quite often expressed in *number of discontinuity surfaces/m*, which results from the fact that the intensity of a discontinuity set is considered to be the number of discontinuity surfaces of that set, which are intersected by a segment with a unit length, and whose orientation is normal to the mean attitude of the considered set.
This second definition only corresponds to the first one if all discontinuity surfaces of the set possess the same attitude (the mean attitude), because only in that case the intersection of any of those discontinuity surfaces with a unit volume having a cylindrical shape, an infinitesimal cross-section, and generatrices which are normal to the mean attitude of the considered discontinuity set (the volume which is equivalent to the segment in the second definition), is equal to the cross--section of that cylindrical volume. It should be noted that it is valid to disregard the occurrence of partial intersections, due to the exiguity of the cross-section of the cylindrical volume.

3.1.2. Spacing

The spacing of a discontinuity set is the inverse of its intensity, i.e., the volume of the rock mass in which the sum of the areas of the discontinuity surfaces of that set, that occur in it, corresponds to a unit area (Grossmann 1967) (Fig. 2).

The spacing should, thus, be expressed in m^3/m^2, although it is usually expressed in *m*, which results from the fact that the spacing of a discontinuity set is considered to be the distance between successive discontinuity surfaces of the set, measured along a straight line with an orientation normal to the mean attitude of the considered set.

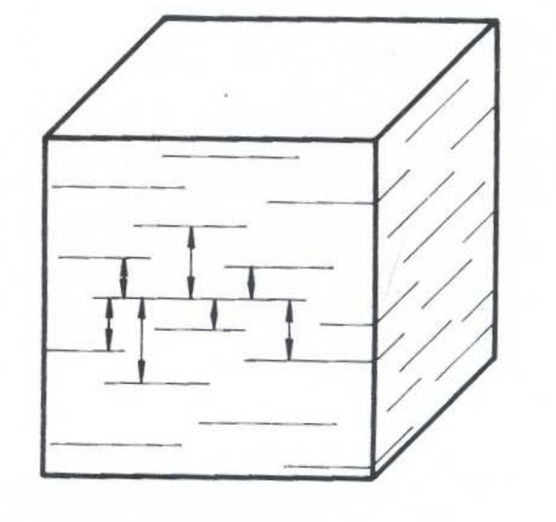

Fig. 2 – Spacing of a discontinuity set

3.2. Determination

As, in general, it is not possible to measure the areas of the jointing surfaces directly, the intensities of the various discontinuity sets occurring in a given rock mass, are determined from the knowledge of the length of the intersections of the different discontinuity surfaces with the observation surface on which the jointing sampling has been performed.

Thus, for a general observation surface (S), the intensity (I) of a chosen discontinuity set is given by (Grossmann 1988)

$$I = \frac{i_t}{\int\int_S \sin\alpha \, dS}$$

as a function of the sum (i_t) of the lengths of the intersections of the discontinuity surfaces of that set with the given observation surface, and the angle (α) between the normal to the surface element (dS) and any normal to the mean attitude of the considered discontinuity set.

In the case of a **plane observation surface**, with an area (S), the last expression reduces to

$$I = \frac{i_t}{\sin a \, S}$$

On the other hand, the general expression can also be transformed, by partial integration, into (Grossmann 1977)

$$I = \frac{i_t}{\int_{h_n} p \, dh_n}$$

as a function of the length (p) of the intersection of the chosen observation surface with the plane possessing the mean attitude of the considered discontinuity set, and corresponding to the length element (dh_n) of the segment (h_n), which is defined on a normal to the mean attitude of that set, by the 2 planes with the mean attitude of the set, which are tangent to the exterior of the observation surface.

In the case of a cylindrical observation surface, for which the length (p) is constant, as, for instance, for a **borehole,** the last expression reduces to

$$I = \frac{i_t}{p\, h_n}$$

If one admits that the intersections of the discontinuity surfaces of the considered set with the cylindrical observation surface have all the same length, i.e., that no partial intersections occur, and that all those discontinuity surfaces have the same attitude (the mean attitude of the set), the last expression can be written as

$$I = \frac{N}{l \cos\varepsilon}$$

as a function of the number (N) of those discontinuity surfaces, the length (l) of any generatrix of the cylindrical observation surface, and the angle (ε) formed by any of those generatrices with the normals to the mean attitude of the considered discontinuity set.
This last expression applies, obviously, also to the cases in which the observation surface reduces to a **segment**, as, for instance, a scanline.
The **spacing** of a discontintuity set is easily calculated from the respective intensity (I), by

$$s = \frac{1}{I}$$

3.3. Statistical Distributions

3.3.1. Poisson distribution

The occurrence of discontinuity surfaces of a same discontinuity set is a phenomenon that, in a homogeneous rock mass, often possesses the following properties:
i) **Stationarity** – the probability that one of those discontinuity surfaces intersects any given segment element (dl) in the rock mass, is approximately equal to (I cos ε dl), as a function of the angle (ε) between the segment element and any normal to the mean attitude of the considered discontinuity set, and the intensity (I) of that set;
ii) **Non-multiplicity –** the probability that more than one discontinuity surfaces of the considered set intersect the aforesaid segment element (dl), is negligible, if compared to (I cos ε dl); and
iii) **Independence –** the number of discontinuity surfaces of the considered set which intersect any given segment in the rock mass, is independent of the number of discontinuity surfaces of that set which intersect any other given segment in the rock mass, as long as the orthogonal projections of those 2 segments on a normal to the mean attitude of the considered set, do not overlap, neither totally, nor partially.
When these conditions are fulfilled, one is in the presence of a Poisson process. (Benjamin & Cornell 1970).

In this case, the probability [P(N)] that any given segment in the rock mass, with a length (l), intersects a number (N) of discontinuity surfaces of the considered set, is given by the Poisson distribution

$$P(N) = \frac{(I\, l \cos\varepsilon)^N\, e^{-I\, l \cos\varepsilon}}{N!}$$

as a function of the intensity (I) of that discontinuity set, and the angle (ε) between the considered segment and any normal to the mean attitude of that set.
Fig. 3 presents some typical cases of the Poisson distribution.

For small values of the parameter (I l cos ε), the Poisson distribution is strongly skewed to the right, but, with the increase of the parameter, this distribution approaches the normal distribution with the mean ($\overline{N}$) and the standard deviation (σ_N).

3.3.2. Exponential distribution

From the expression of the Poisson distribution, one deduces directly that the probability [P(0)] that any given segment in the rock mass, with a length (l), does not intersect any discontinuity surface of the considered set, i.e., that the distance between any 2 successive intersections of the discontinuity surfaces of that set with any given straight line in the rock mass, is not less than (l), is given by

$$P(0) = e^{-I\,l\cos\varepsilon}$$

as a function of the intensity (I) of the considered discontinuity set, and the angle (ε) between either the considered segment, or the considered straight line, and any normal to the mean attitude of that set.

By derivation of the last expression, one obtains the probability density function [f(l)] corresponding to a distance (l) between 2 successive intersections of the discontinuity surfaces of the considered set with any given straight line in the rock mass, i.e., in a certain sense, the probability density function of a "spacing", which is given by

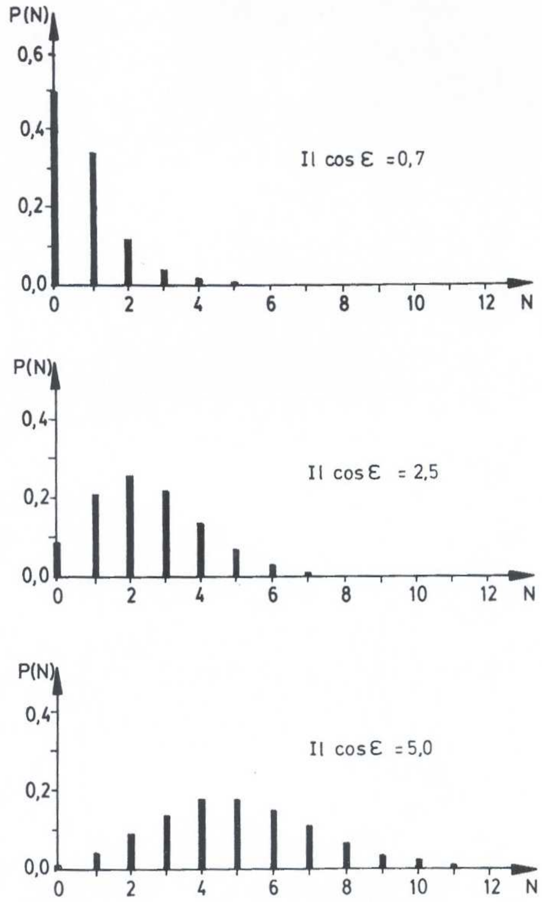

Fig. 3 – Poisson distribution –Variation of the probability with the number of discontinuity surfaces, for 3 values of the parameter (I l cos ε)

$$f(l) = I\cos\varepsilon\; e^{-I\,l\cos\varepsilon}$$

This expression corresponds to an exponential distribution with the parameter (I cos ε).

3.3.3. Gamma distribution

The probability [$P_N(l)$] that the intersection of order (N) of any given straight line in the rock mass, with the discontinuity surfaces of the considered set lies at a distance not less than (l) from any given point on that straight line, can also be deduced from the expression of the Poisson distribution, and is given by

$$P_N(l) = \sum_{i=0}^{i=N-1} \frac{(I\,l\cos\varepsilon)^i\, e^{-I\,l\cos\varepsilon}}{i!} = \frac{\Gamma(N,\, I\,l\cos\varepsilon)}{\Gamma(N)}$$

as a function of the intensity (I) of that discontinuity set, the angle (ε) between the considered straight line and any normal to the mean attitude of that set, the gamma function [Γ(N)] with the parameter (N), and the incomplete gamma function [Γ(N, x)] with the parameters (N) and (x).

By derivation of the expression of [$P_N(l)$], one obtains the probability density function [$f_N(l)$] corresponding to a distance (l) between the point chosen on the straight line and the intersection of order (N) of that straight line with the discontinuity surfaces of the considered set, which is given by

$$f_N(l) = \frac{(I\, l \cos\, \varepsilon)^N\, e^{-I\, l \cos\, \varepsilon}}{l\, \Gamma(N)}$$

This distribution is a gamma distribution with the parameters (N) and (I cos ε) (if (N) takes only integer values, it is also called an Erlang distribution).

Fig. 4 presents some examples of the Erlang (gamma) distribution.

For small values of the parameter (N), the gamma distribution is strongly skewed to the right, but, with the increase of the parameter, this distribution approaches the normal distribution with the mean ($\overline{l_N}$) and the standard deviation (σ_{l_N}).

It should be noted that the aforementioned exponential distribution corresponds to the particular case of the gamma distribution in which the parameter (N) takes the value 1.

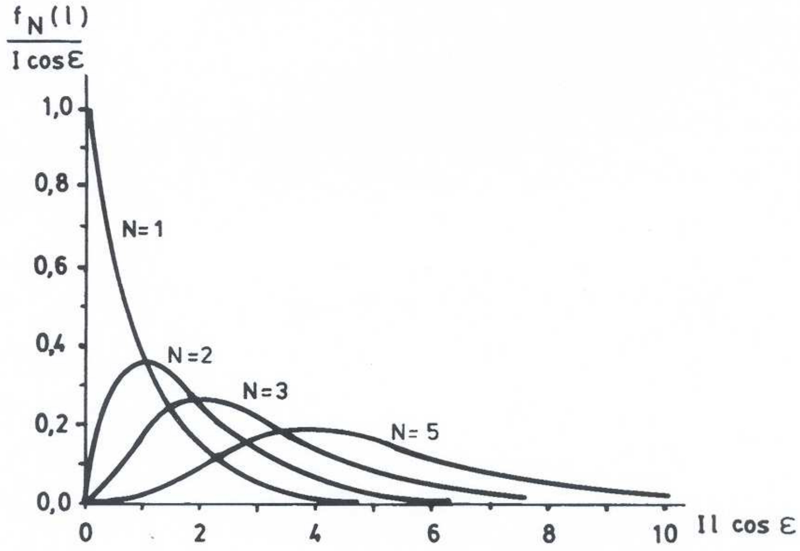

Fig. 4 – Erlang (gamma) distribution –Variation of the probability density with the variable (I l cos ε), for 4 values of the parameter (N)

3.3.4. Lognormal distribution

It is found, sometimes, that some discontinuity sets occurring in homogeneous rock masses, are not correctly described by means of a Poisson process. This fact may, for instance, be due to the way the rock was formed, to the existence of discontinuity sets generated before the origin of the discontinuity sets which are not susceptible to be described by means of a Poisson model, etc.

In these cases, a description with the help of a lognormal distribution has been applied with success.

The probability [P(0)] that any given segment in the rock mass, with a length (l), does not intersect any discontinuity surface of the considered set, i.e., that the distance between any 2 successive intersections of the discontinuity surfaces of that set with any given straight line in the rock mass, is not less than (l), is, then, given by (Grossmann 1988)

$$P(0) = Q\left[\frac{1}{\sigma}\ln\,(I\, l \cos\, \varepsilon) + \frac{\sigma}{2}\right]$$

as a function of the intensity (I) of the considered discontinuity set, the angle (ε) between either the considered segment, or the considered straight line and any normal to the mean attitude of that set, the (unitless) standard deviation (σ) of the distribution, and the upper tail area [Q(x)] of the standardized normal (Gaussian) distribution, for the value (x) of the standardized variable.

The probability density function [f(l)] corresponding to a distance (l) between 2 successive intersections of the discontinuity surfaces of the considered set with any given straight line in the rock mass, i.e., in a certain sense, the probability density function of a “spacing”, is given by the lognormal distribution

$$f(l) = \frac{1}{\sqrt{2\pi}\,\sigma\, l}\, e^{-\frac{1}{2}\left[\frac{1}{\sigma}\ln(I\, l\cos\varepsilon)+\frac{\sigma}{2}\right]^2}$$

For small values of the parameter (σ), the lognormal distribution approaches a normal distribution, but, for the values of (σ) usually occurring in the practice (always greater than 0,7, and, generally, even greater than 1,5), this distribution is strongly skewed to the right.

3.3.5. Comments

Independently of the applicable statistical model, we may, thus, state that the distribution of the distance between 2 successive intersections of the discontinuity surfaces of a given set with any given straight line in the rock mass, i.e., in a certain sense, the distribution of a "spacing", has a positive coefficient of skewness, its mode being lower than its mean.
In consequence, we may also state that, independently of the applicable statistical model, the distribution of the probability [P(N)] that any given segment in the rock mass, with a length (l), intersects (N) discontinuity surfaces of the considered set, can only be assimilated to a normal (Gaussian) distribution when the length (l) is sufficiently great (in view of the intensity of the discontinuity set, and taking into account the angle between the chosen segment and any normal to the mean attitude of the discontinuity set), so that the mean of the distribution of [P(N)] has a minimum value of 9.
As, however, the scale of many engineering problems implies the consideration of segments for which the above-mentioned condition is not fulfilled, the distribution of [P(N)] will, usually, also have a positive coefficient of skewness, its mode being lower than its mean.
In short, care must be taken with the fact that, nearly always, the means of the different characteristics related to the intensity of the discontinuity sets, do not correspond neither to the most frequent values (modes), nor to the middle values (medians) of the respective distribution.

3.3.6. Example

The recognition of the skewed character of the distributions of many practical parameters connected with the intensities of the discontinuity sets, has still not entered into the domain of Rock Mechanics' common knowledge.
Due to this fact, the foreseeable occurrence of restricted zones with a great number of discontinuity surfaces of a given set, tends always to be considered as an "unpredictable abnormality".
For instance, if a tunnel, in a homogeneous rock mass, runs normal to the mean attitude of a joint set whose intensity is 1 m^2/m^3, and for which the occurrence of the joints can be described by a Poisson model, the probability that a certain number of joints of that set occurs in any given 10 m stretch of the tunnel, is shown in Table I.
The analysis of this table indicates that, in each km of that tunnel, one may expect one 10 m stretch with 3 or less joints of the considered set, but also another 10 m stretch with 18 or more joints of that set, a more than sixfold increase in the number of joints.

Number of joints per 10 m stretch	**Probability (%)**
< 4	1,0
4	1,9
5	3,8
6	6,3
7	9,0
8	11,3
9	12,5
10	12,5
11	11,4
12	9,5
13	7,3
14	5,2
15	3,5
16	2,2
17	1,3
> 17	1,4

Table I – Probability of occurrence of joints of a set with an intensity of 1 m^2/m^3, in a 10 m.stretch of a tunnel running normal to the mean attitude of the set (Grossmann 1988)

Fig. 5 presents the result of a computer simulation of the aforesaid occurrence of a joint set whose intensity is 1 m^2/m^3, in a tunnel with a length of 1 km, clearly showing that, in a **homogeneous rock mass**, there are zones which an unexperienced observer would classify as totally different in their jointing.

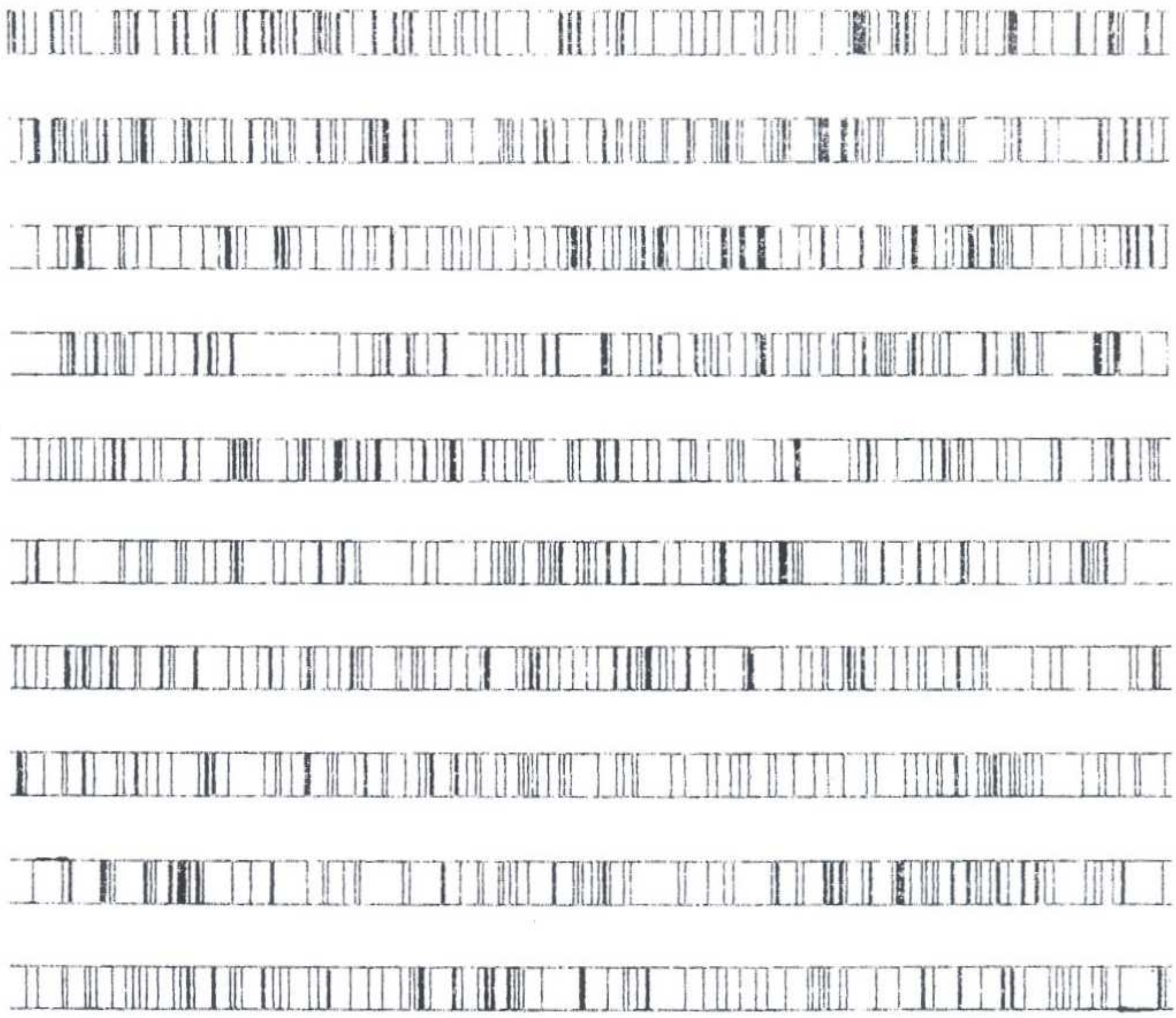

Fig. 5 – Tunnel in a homogeneous rock mass – Intersections of the joints of a set which can be described by a Poisson model

3.4. Volume of the Blocks

3.4.1. Premises

The knowledge of the volume of the blocks which are defined by the jointing of a rock mass, can have a great practical interest, because, in many situations, the use of those blocks is conditioned by a minimum and/or maximum volume.

That knowledge can easily be obtained, if the following 3 hypotheses are assumed:

i) the rock mass presents only 3 discontinuity sets;

ii) all the discontinuity surfaces of a same set possess the same attitude (the mean attitude of the discontinuity set); and

iii) all discontinuity surfaces end at other discontinuity surfaces.
The 1st hypothesis will, as a rule, be easily accepted in the case of rock masses formed by anisotropic rocks, since the experience has shown that their discontinuity system is, very often, basically constituted by 3 approximately triorthogonal discontinuity sets.
For the rock masses with more than 3 important discontinuity sets, one would have to merge, if possible, groups of neighbouring sets into single sets.
The 3rd hypothesis, although seeming very restrictive, is based on the practice. In fact, it has been verified that, as a rule, only a small percentage of discontinuity surfaces seem to end in other ways than at other discontinuity surfaces (for instance, Kikuchi et al. (1985) indicate 11,5 %), and it can always be argued that, possibly, some of those discontinuity surfaces still continue until they find others, but with such a tiny opening, that they can not be detected by the employed observation techniques.
If all the 3 aforesaid simplifying hypotheses are accepted, all the blocks of the given rock mass have the shape of parallelepipeds, the attitudes of their 3 pairs of parallel faces being, respectively, the mean attitudes of the 3 considered discontinuity sets
The volume (V) of a parallelepiped whose 3 pairs of parallel faces have, respectively, the attitudes (A), (B), and (C), is given by

$$V = \frac{d_A\, d_B\, d_C}{\sqrt{1 - \cos^2 \alpha_{BC} - \cos^2 \alpha_{CA} - \cos^2 \alpha_{AB} + 2 \cos \alpha_{BC} \cos \alpha_{CA} \cos \alpha_{AB}}}$$

as a function of the distances (d_A), (d_B), and (d_C) between the 2 faces with, respectively, the attitudes (A), (B), and (C), and the angles (α_{BC}), (α_{CA}), and (σ_{AB}) between, respectively, the attitudes (B) and (C), (C) and (A), and (A) and (B).

3.4.2. Statistical distribution

When the occurrence of the discontinuity surfaces of the 3 chosen sets can be described by means of **Poisson** processes, the probability density function [f(V)] of the volume (V) of the blocks of the rock mass is given by (Grossmann 1986)

$$f(V) = \frac{4}{\overline{V}} \int_0^\infty \frac{1}{t} K_0(t)\, e^{-\frac{4V}{\overline{V} t^2}}\, dt$$

as a function of the modified Bessel function of the 2nd kind and order 0 [$K_0(t)$], with the argument (t), and the mean of the distribution ($\overline{V}$), which is calculated by

$$\overline{V} = \frac{1}{I_A\, I_B\, I_C \sqrt{1 - \cos^2 \alpha_{BC} - \cos^2 \alpha_{CA} - \cos^2 \alpha_{AB} + 2 \cos\alpha_{BC} \cos\alpha_{CA} \cos\alpha_{AB}}}$$

as a function of the intensities (I_A), (I_B), and (I_C) of the 3 discontinuity sets, with, respectively, the mean attitudes (A), (B), and (C), and the angles (α_{BC}), (α_{CA}), and (α_{AB}) between, respectively, the mean attitudes (B) and (C), (C) and (A), and (A) and (B).
In this case, the distribution of the volume of the blocks of the rock mass has the standard deviation (σ_V), given by

$$\sigma_V = \sqrt{7}\, \overline{V}$$

The cumulative distribution function [F(V)] of the volume of the blocks of the rock mass is, then, given by

$$F(V) = 1 - \int_0^\infty t\, K_0(t)\, e^{-\frac{4V}{\overline{V}t^2}}\, dt$$

and the fraction [p(V)] of the rock mass, constituted by the blocks whose individual volume does not exceed (V), by (Grossmann 1986)

$$p(V) = 1 - \int_0^\infty t\, K_0(t) \left(\frac{V}{\overline{V}} + \frac{t^2}{4} \right) e^{-\frac{4V}{\overline{V}t^2}}\, dt$$

Fig. 6 presents the probability density function and the cumulative distribution function of the volume of the blocks of a rock mass, for which the 3 pertinent discontinuity sets can be described by Poisson models. In this case, the probability density function is a monotonically decreasing function of the block volume.

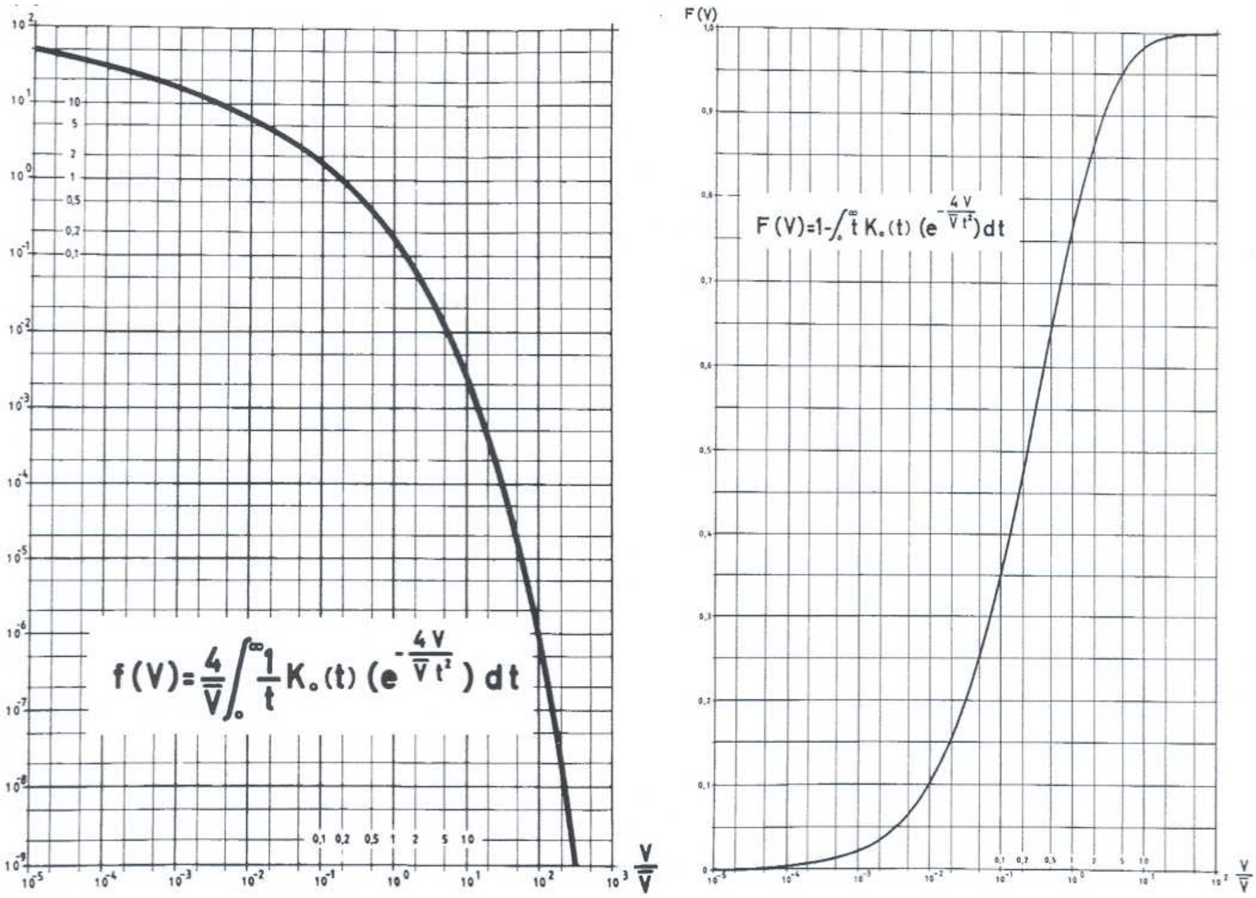

Fig. 6 – Probability density function (left) and cumulative distribution function (right) of the volume of the blocks of a rock mass, when the discontinuity sets are described by Poisson models

However, if the occurrence of the discontinuity surfaces of the 3 chosen sets is described with the help of **lognormal** distributions, the volume (V) of the blocks of the rock mass follows also a lognormal distribution, whose probability density function [f(V)] is given by

$$f(V) = \frac{1}{\sqrt{2\pi}\,\sigma V}\, e^{-\frac{1}{2\sigma^2}\left(\ln\frac{V}{\xi}\right)^2}$$

as a function of the 2 parameters of the distribution (the median (ξ) and the standard deviation (σ)), which are calculated by

$$\xi = \frac{1}{I_A\, I_B\, I_C\, e^{\frac{\sigma^2}{2}} \sqrt{1 - \cos^2 \alpha_{BC} - \cos^2 \alpha_{CA} - \cos^2 \alpha_{AB} + 2\cos\alpha_{BC}\cos\alpha_{CA}\cos\alpha_{AB}}}$$

as a function of the intensities (I_A), (I_B), and (I_C) of the 3 discontinuity sets, with, respectively, the mean attitudes (A), (B), and (C), and the angles (α_{BC}), (α_{CA}), and (α_{AB}) between, respectively, the mean attitudes (B) and (C), (C) and (A), and (A) and (B); and

$$\sigma = \sqrt{\sigma_A^2 + \sigma_B^2 + \sigma_C^2}$$

as a function of the standard deviations (σ_A), (σ_B), and (σ_C) of the distributions of the intersections with a straight line, of the discontinuity surfaces of the 3 considered sets, with, respectively, the mean attitudes (A), (B), and (C).

The cumulative distribution function [F(V)] of the volume of the blocks of the rock mass is, in this case, given by

$$F(V) = P\left(\frac{1}{\sigma}\ln\frac{V}{\overline{\overline{V}}} + \frac{\sigma}{2}\right)$$

as a function of the lower tail area of the standardized normal distribution [P(x)], for the value (x) of the standardized variable; and the fraction [p(V)] of the rock mass, made up by the blocks whose individual volume does not exceed (V), is given by

$$p(V) = P\left(\frac{1}{\sigma}\ln\frac{V}{\overline{\overline{V}}} - \frac{\sigma}{2}\right)$$

Fig. 7 presents an example of a graphical solution (with the help of a lognormal probability paper) of the last equation.

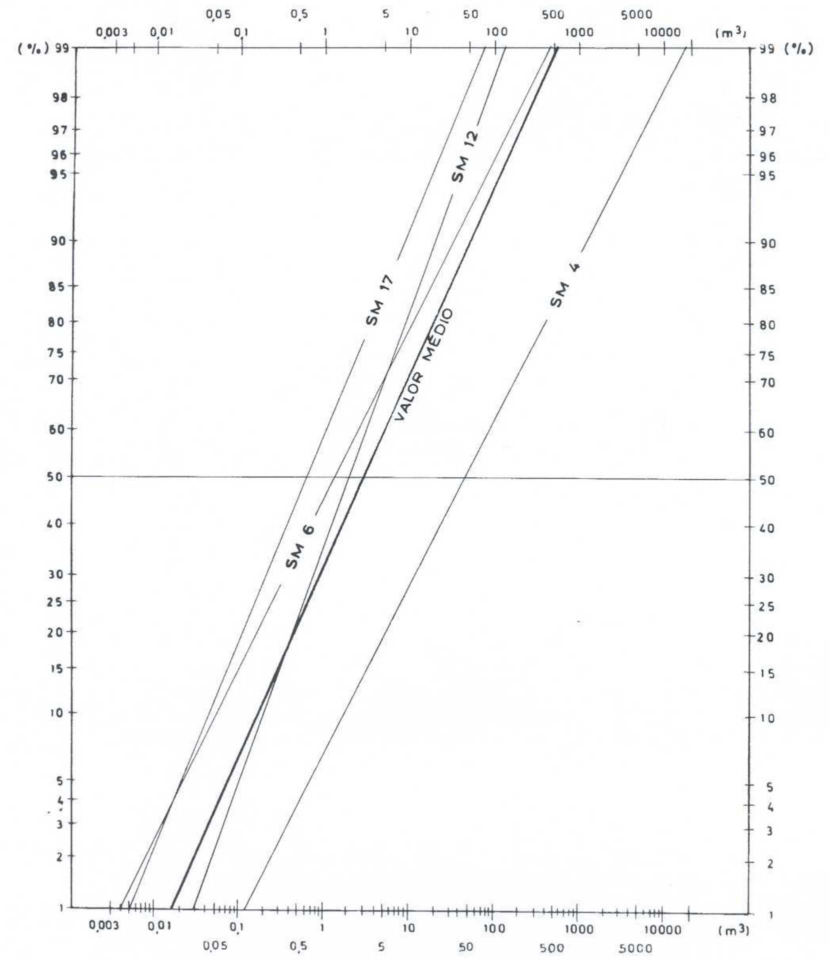

Fig. 7 – Fraction of the rock mass, whose individual blocks do not exceed a given volume, for a lognormal block volume distribution (dioritic rock mass (LNEC 1973 a))

4. AREA OF THE DISCONTINUITY SETS

4.1. Concepts

4.1.1. Area

The parameter area describes the size of the discontinuity surfaces of the set, independently of their shape.

4.1.2. Equivalent radius

The equivalent radius (Grossmann 1984) of a discontinuity surface is the radius of the circle whose area is equal to that of the discontinuity surface.

4.2. Determination

As said before, in general, it is not possible to measure directly the areas of the jointing surfaces occurring in a given rock mass.
Therefore, as for the intensity, one resorts to the knowledge of the lengths of the intersections of the different discontinuity surfaces with the observation surface on which the jointing sampling has been performed, in order to obtain an information about the area of the discontinuity surfaces of the different sets occurring in the rock mass.
Unlike what happens with the expressions allowing to obtain the intensity of a discontinuity set, the expressions giving the mean area of the discontinuity surfaces of a set, depend on the type of the distribution of the areas of the jointing surfaces of the set, through the parameter (k) which relates the mean equivalent radius ($\overline{R}$) of the discontinuity surfaces of that set to the mean area ($\overline{A}$) of those surfaces, in the equality (Grossmann 1987)

$$\overline{R} = k\sqrt{\overline{A}}$$

Thus, for a **convex, closed observation surface** (S), delimiting the volume (V), for which the normal to the surface element (dS) forms an angle (α) with any normal to the mean attitude of the considered discontinuity set, the mean area of the discontinuity surfaces of that set is given by (Grossmann 1987)

$$\overline{A} = \left[\frac{\dfrac{k}{2} + \sqrt{\dfrac{k^2}{4} + \dfrac{V - V'}{\iint_S \sin\alpha \; dS}\left(\dfrac{1}{\overline{i}} - \dfrac{h_N}{\iint_S \sin\alpha \; dS}\right)}}{\dfrac{1}{\overline{i}} - \dfrac{h_N}{\iint_S \sin\alpha \; dS}} \right]^2$$

as a function of the mean length ($\overline{i}$) of the intersections of the discontinuity surfaces of the considered set with the observation surface, the distance (h_N) between the 2 planes with the mean attitude of that set, which are tangent to the observation surface, and a corrective term (V’) of the volume (V), which complies with the inequalities

$$0 \leq V' \leq V$$

and depends on the shape of the observation surface and on the type of the distribution of the areas of the discontinuity surfaces of the considered set.
For the case of a **plane convex observation surface**, with an area (S), a transformation of the last equality for a prism with a height 0 and bases with the area (S), gives (Grossmann 1977)

$$\bar{A} = \left(\frac{2k}{\frac{1}{\bar{i}} - \frac{h_N}{\sin\alpha\, S}} \right)^2$$

4.3. Statistical Distribution

4.3.1. Bessel function distribution

As has been referred to, while presenting the basic hypotheses for the evaluation of the volume of the blocks of a rock mass, the experience has shown that the majority of the jointing surfaces occurring in a rock mass, seem to end at other jointing surfaces.
This fact implies that those discontinuity surfaces have a polygonal shape, and, thus, their area depends, basically, on the product of 2 distances between opposite sides of the polygon. These distances, however, correspond to distances between successive discontinuity surfaces along straight lines, which, in general, follow exponential distributions (Priest & Hudson, 1976). The distribution of the areas of the discontinuity surfaces can, then, be obtained by multiplying 2 exponential distributions, the result being a Bessel function distribution (Grossmann 1986).
Moreover, a study of 10 different rock masses (Hudson & Priest, 1979) showed a good agreement between the values measured for the areas of the different discontinuity surfaces, and the corresponding Bessel function distributions.
The probability density function [f(A)] of the area (A) of the discontinuity surfaces of a set presenting a Bessel function distribution, is given by (Grossmann & Muralha 1987)

$$f(A) = \frac{2}{\bar{A}} K_0\left(2\sqrt{\frac{A}{\bar{A}}}\right)$$

as a function of the modified Bessel function of the 2nd kind and order (n) [$K_n(x)$], with the argument (x), and the mean of the distribution ($\bar{A}$).
In this case, the distribution of the area of the discontinuity surfaces of the set has the standard deviation (σ_A), given by

$$\sigma_A = \sqrt{3}\,\bar{A}$$

The cumulative distribution function [F(A)] of the area of the discontinuity surfaces of a set is, then, given by

$$F(A) = 1 - 2\sqrt{\frac{A}{\bar{A}}}\,K_1\left(2\sqrt{\frac{A}{\bar{A}}}\right)$$

and the contribution [p(A)] of the discontinuity surfaces of the considered set, whose individual area does not exceed (A), to the total fracturing which that set induces in the rock mass, given by (Grossmann 1991)

$$p(A) = 1 - 2\frac{A}{\bar{A}}\left[K_2\left(2\sqrt{\frac{A}{\bar{A}}}\right) + \sqrt{\frac{\bar{A}}{A}}\,K_1\left(2\sqrt{\frac{A}{\bar{A}}}\right)\right]$$

Fig. 8 presents the probability density function for the Bessel function distribution.
In this case, the probability density function has a vertical asymptote at the origin, and decreases monotonically with the area, tending to 0.

For the Bessel function distribution, the value of the parameter (k) referred to above, is given by

$$k = \frac{\sqrt{\pi}}{4}$$

4.3.2. Lognormal distribution

However, there are also cases reported in the literature (Piteau 1973), in which it has been verified that the dimensions of the discontinuity surfaces in the directions of the strike and of the dip, follow lognormal distributions.
In those cases, also the areas of the discontinuity surfaces can be described by a lognormal distribution.
The probability density function [f(A)] of the area (A) of the discontinuity surfaces of a set presenting a lognormal distribution, is given by

$$f(A) = \frac{1}{\sqrt{2\pi}\,\sigma A}\, e^{-\frac{1}{2\sigma^2}\left(\ln\frac{A}{\xi}\right)^2}$$

as a function of the 2 parameters of the distribution (the median (ξ) and the standard deviation (σ)).

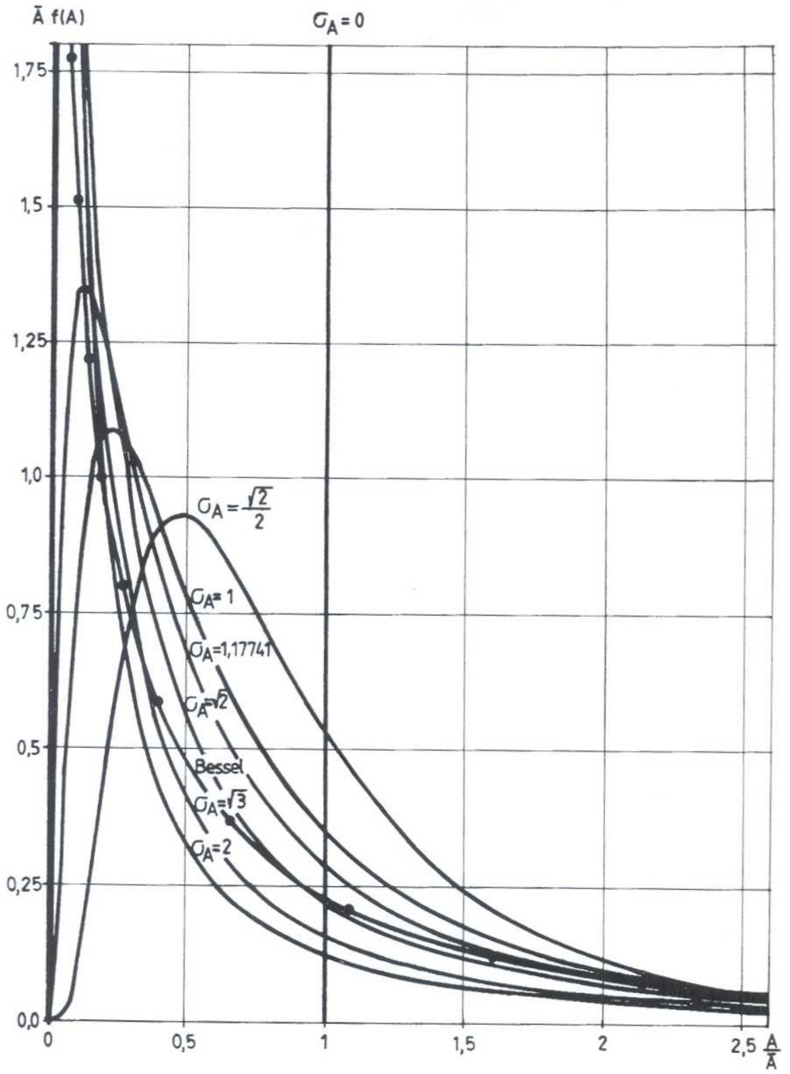

Fig. 8 – Probability density function of the area of a discontinuity set – Bessel function distribution (line with dots) and 7 lognormal distributions with different standard deviations, for the same mean area

The mean area ($\overline{A}$) is, in this case, given by

$$\overline{A} = \xi\, e^{\frac{\sigma^2}{2}}$$

and the standard deviation of the areas (σ_A) (which is different from the parameter (σ) of the distribution) by

$$\sigma_A = \overline{A}\sqrt{e^{\sigma^2} - 1}$$

The cumulative distribution function [F(A)] of the area of the discontinuity surfaces of a set is, then, given by

$$F(A) = P\left(\frac{1}{\sigma}\ln\frac{A}{\xi}\right)$$

as a function of the lower tail area of the standardized normal distribution [P(x)], for the value (x) of the standardized variable; and the contribution [p(A)] of the discontinuity surfaces of the considered set, whose individual area does not exceed (A), to the total fracturing which that set induces in the rock mass, by (Grossmann 1991)

$$p(A) = P\left(\frac{1}{\sigma}\ln\frac{A}{\xi} - \sigma\right)$$

Fig. 8 also presents the probability density functions for 7 different lognormal distributions, which have all the same mean area as the shown Bessel function distribution.
For the lognormal distribution, the value of the parameter (k), defined as stated before, is given by

$$k = \frac{1}{\sqrt{\pi\, e^{\frac{\sigma^2}{4}}}}$$

4.3.3. Comments

Independently of its type, the distribution of the areas of the discontinuity surfaces of a set is a distribution with a strong positive asymmetry, for which both the most frequent value (mode) and the middle value (median) are lower than its mean.
On the other hand, the intersection of a discontinuity surface with the possible observation surface of the rock mass, only once in a while corresponds to the maximum dimension of the discontinuity surface, even in those cases in which the observation surface is large in relation to the area of that discontinuity surface.
From all this, it results that, usually, an unexperienced observer will be induced to underestimate the mean areas of the discontinuity surfaces of the different sets occurring in a rock mass, and so, in general, he/she will not be on the safe side.

5. APERTURE OF THE DISCONTINUITY SETS

5.1. Concept

The parameter aperture describes a given dimension normal to the discontinuity surfaces of a set, which is chosen according to the necessities of the study under consideration, and to the type of geological feature of those surfaces.
Thus, in the case of fissures, fractures, or joints, the aperture corresponds, usually, to the distance between the faces of the 2 blocks contiguous to the considered jointing surface; in the case of veinlets, veins, or faults, the aperture is equivalent to the thickness of the respective filling; in other cases, still, the aperture designates the transversal dimension of the whole zone of altered rock, which accompanies certain types of discontinuity surfaces of the rock masses.
When the model adopted for the discontinuity surfaces, is the one of prisms with a very small height in relation to the dimension of its bases, obviously, each discontinuity surface will be characterized exclusively by 1 aperture value, which, usually, will be the mean value of the apertures determined at different points of the discontinuity surface.
This model fully satisfies when the discontinuity surfaces are plane, or, else, when the coefficient of variation of the apertures determined at the different points of the discontinuity surface, is small.
For many jointing surfaces occurring in the rock masses, however, the above premises are not verified, and, thus, their aperture requires a more elaborate description.

5.2. Statistical Distribution

The mathematical model for the distribution function of the apertures of the discontinuity surfaces of a set presented in the literature, assumes that the discontinuity surfaces are prisms with a very small height in relation to the dimension of its bases, i.e., it is a model for the distribution function of the mean apertures of the discontinuity surfaces of a set.
It is again a lognormal distribution, and, thus, the probability density function [f(a)] of the (mean) aperture (a) of the discontinuity surfaces of a set is given by

$$f(a) = \frac{1}{\sqrt{2\pi}\,\sigma\, a}\; e^{-\frac{1}{2\sigma^2}\left(\ln\frac{a}{\xi}\right)^2}$$

as a function of the 2 parameters of the distribution (the median (ξ) and the standard deviation (σ)).
The distribution of the (mean) apertures of the discontinuity surfaces of a set is, still, a distribution with a positive asymmetry, for which both the most frequent value (mode) and the middle value (median) are lower than its mean.

5.3. Roughness and waviness

As already said, the discontinuity surfaces can, in many cases, be modelled as prisms with a very small height in relation to the dimension of its bases. When, however, this simplification is not acceptable, the deviations between the discontinuity surfaces and the respective mean planes are characterized by the 2 parameters roughness and waviness.
For the description of those deviations, one resorts, usually, to the uni- or bidimensional harmonic analysis, i.e., the deviations are interpreted as a superposition of several simple sinusoidal phenomena, each one possessing its spatial period and its amplitude.
The roughness concerns those components of the general undulatory phenomenon, whose spatial period is small, at most, of the order of magnitude of the size of the crystals in the rock, while the term waviness is applied to the components of the general undulatory phenomenon with a greater spatial period.
The literature (Piteau 1973; Greenwood et al. 1984) refers also to criteria for the differentiation between the roughness and the waviness, which are based on the characteristics related to the behaviour of the irregularities of the discontinuity surface under shear or compression loads.
These criteria, whose use is not easier that the use of the above-mentioned geometric criterium, present the disadvantage that they can lead to different results for the same discontinuity surface, when different modes of applying the loads, different load levels, etc. are used.

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