ELASTIC DESIGN OF TAPERED BEAM-COLUMNS SUBJECTED TO CONCENTRATED AXIAL AND TRANSVERSAL LOADS

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ABSTRACT

Tapered members are an efficient solution for steel beams, allowing an adjustment of the cross-sections resistance to the applied loads. However, while the critical cross-section of uniform beams is always the one subjected to the most unfavourable load combination, it is not the case with tapered beams, since both acting and resistant values of the cross section internal loads vary along the beam length. So, one of the main problems in the design of these structural members lies on the determination of the beam critical cross-section where yielding occurs for the first time.

This paper presents a method of calculation of analytical expressions for the elastic design of tapered beams subjected to bending and axial force. These relationships allow the beam critical section, its internal forces and the maximum loads carried by the beam at its elastic limit state, to be determined. Some examples are presented to show the possibilities of this proposal.

Key Words: *Steel structures, Elastic design, Tapered beam-columns, Bending and Compression, Tapered beam critical cross-section.*

1. INTRODUCTION

Tapered structural members represent generally a valuable solution for steel beams, since the adjustment of the cross-section resistance to the applied loads is possible. The variations of the beam cross-sections following those of the applied loads allow the beam self-weight to be optimised. So, that leads to a cheaper solution, as far as the weight of steel required for its fabrication is concerned.

The cross-section where the yield stress is reached for the first time, in its most strained fibres, is defined as the beam critical section. The elastic limit of the beam corresponds to the elastic limit of its critical cross-section.

If the critical section of uniform beams is always the one subjected to the most unfavourable load combination, it is no longer the case with tapered members, since both acting and resistant values of the cross-section internal loads vary along the beam length.

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Therefore, one of the major difficulties in the design of non-uniform structural members lies on the determination of the beam critical section, which controls the elastic resistance of the entire element.

This paper presents a method of deduction of analytical expressions for the elastic design of tapered beams submitted to bending and axial force, and not subjected to any kind of buckling.

These expressions allow the beam critical section, its internal forces and the maximum loads, carried by the beam at its elastic limit state, to be found. So, it is possible to compare the reduction of the tapered beam self-weight, when compared to a uniform beam with the same elastic limit state.

The expressions are written as a function of non-dimensional parameters, which allow the influence of the relative variations in the beam geometry or loading parameters over the tapered beam resistance to be analysed [1, 2].

In order to help the understanding of this method, examples of application are presented for a simple case of a tapered cantilever beam with a rectangular cross-section, subjected to concentrated axial and transversal loads at its free end.

The application of the method to other cases of tapered beams, with different cross-sections or other types of loading is made the same way, after defining the respective distributions of acting and resisting internal forces. The distributions of these resisting internal forces are also presented for the particular case of a tapered beam with I cross-sections, but the application of the method to this types of beams is not presented, since it exceeds the limits of this paper.

2. DISTRIBUTION OF THE RESISTING CROSS-SECTION INTERNAL FORCES

2.1 Beams with rectangular cross-sections

The following relationships apply to beams with rectangular cross-sections with a constant width b and a linearly varying height $h(x)$, between two extreme values h (minimum) and H (maximum) located at the ends of the beam (fig. 1). The beam length is designated by L.

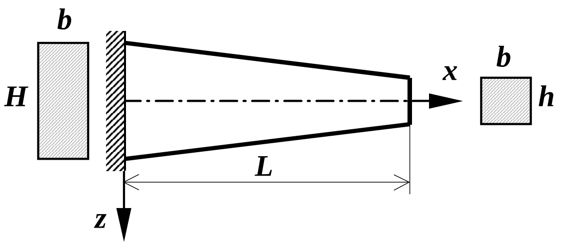

Fig. 1 – Dimensional variables

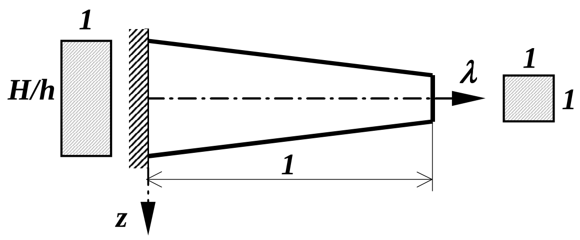

Fig. 2 – Reduced variables

The analytical expressions resulting from the application of the method are written in function of reduced (non-dimensional) variables [2] such as, for instance:

$$\lambda = x/L \quad (1) \qquad \alpha_v = 1 - h/H \quad (2)$$

Therefore, the height of a cross-section with a reduced co-ordinate λ is given by eq. (3) and the area and the second moment of inertia of this cross-section are given by eq. (4) and (5):

$$h(\lambda) = \frac{1-\alpha_v \lambda}{1-\alpha_v}\, h \quad (3)$$

$$A(\lambda) = bh(\lambda) = \frac{1-\alpha_v \lambda}{1-\alpha_v}\, bh = \frac{1-\alpha_v \lambda}{1-\alpha_v}\, A_{min} \quad (4)$$

$$I(\lambda)=\frac{bh(\lambda)^3}{12}=\left(\frac{1-\alpha_v\,\lambda}{1-\alpha_v}\right)^3\frac{bh^3}{12}=\left(\frac{1-\alpha_v\,\lambda}{1-\alpha_v}\right)^3 I_{min} \tag{5}$$

The resisting axial force $N_y(\lambda)$ and bending moment $M_y(\lambda)$ of this section correspond to the maximum internal forces that the cross-section is able to support in the elastic domain. They are defined by eq. (6) and (7), where ε_y represents the elastic limit strain of the material.

$$N_y(\lambda)=EA(\lambda)\varepsilon_y=E\,b\,h\left(\frac{1-\alpha_v\,\lambda}{1-\alpha_v}\right)\varepsilon_y=\left(\frac{1-\alpha_v\,\lambda}{1-\alpha_v}\right)N_{y.min} \tag{6}$$

$$M_y(\lambda)=\frac{I(\lambda)\,\sigma_y}{h(\lambda)/2}=\left(\frac{1-\alpha_v\,\lambda}{1-\alpha_v}\right)^3 I_{min}\,\frac{E\varepsilon_y}{h(\lambda)/2}=\left(\frac{1-\alpha_v\,\lambda}{1-\alpha_v}\right)^2\frac{EI_{min}\varepsilon_y}{h/2}=\left(\frac{1-\alpha_v\,\lambda}{1-\alpha_v}\right)^2 M_{y.min} \tag{7}$$

The resisting internal forces $N_y(\lambda)$ and $M_y(\lambda)$ may be written in a non-dimensional form, according to eq. (8) and (9):

$$n_y(\lambda)=\frac{N_y(\lambda)}{N_{y.min}}=\left(\frac{1-\alpha_v\,\lambda}{1-\alpha_v}\right) \tag{8}$$

$$m_y(\lambda)=\frac{M_y(\lambda)}{M_{y.min}}=\left(\frac{1-\alpha_v\,\lambda}{1-\alpha_v}\right)^2 \tag{9}$$

2.2 Beams with I-shaped cross-sections

The following expressions apply to beams with I-shaped cross-sections with constant width b, and with a height $h(x)$ varying linearly between two extreme values h (minimum) and H (maximum) located at the beam ends (fig. 3). The flange and web thickness, t_f and t_w, are expected to be constant along the length of the beam, L.

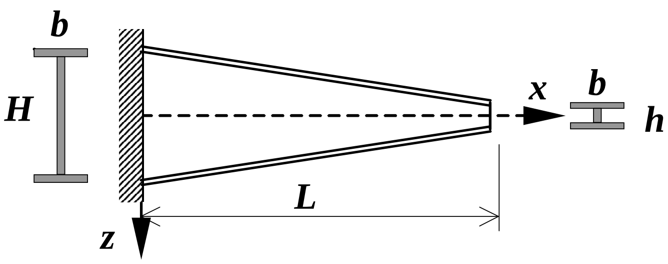

Fig. 3 – Tapered I-shaped beam

The shape of the smallest cross-section is characterised by the reduced parameters α_b and α_h:

$$\alpha_b=1-\frac{t_w}{b} \tag{10}$$

$$\alpha_h=1-\frac{2\,t_f}{h} \tag{11}$$

Since the height of the cross-section varies along the beam length, the α_h parameter of a cross-section with a reduced co-ordinate λ is equal to [2]:

$$\alpha_{h(\lambda)}=1-\frac{2\,t_f}{h(\lambda)}=1-\frac{2\,t_f}{h}\,\frac{1-\alpha_v}{1-\alpha_v\,\lambda}=1-\frac{(1-\alpha_h)(1-\alpha_v)}{1-\alpha_v\,\lambda} \tag{12}$$

In the particular case of the largest cross-section:

$$\alpha_{h(\lambda=0)}=\alpha_H=1-\frac{2\,t_f}{H}=1-(1-\alpha_h)(1-\alpha_v) \tag{13}$$

The area $A(\lambda)$ and the second moment of inertia $I(\lambda)$ are given by eq. (14) and (15), [2, 3], and the resisting axial force $N_y(\lambda)$ and bending moment $M_y(\lambda)$ are defined by eq. (16) and (17):

$$A(\lambda)=\left(1-\alpha_b\alpha_{h(\lambda)}\right)bh(\lambda)=\left(1-\alpha_b\left(1-\frac{(1-\alpha_h)(1-\alpha_v)}{1-\alpha_v\ \lambda}\right)\right)\frac{1-\alpha_v\ \lambda}{1-\alpha_v}\ \frac{A_{min}}{1-\alpha_b\alpha_h} \tag{14}$$

$$I(\lambda)=\left(1-\alpha_b\left(1-\frac{(1-\alpha_h)(1-\alpha_v)}{1-\alpha_v\ \lambda}\right)^3\right)\frac{bh(\lambda)^3}{12}=\frac{(1-\alpha_v\ \lambda)^3-\alpha_b(\alpha_H-\alpha_v\ \lambda)^3}{(1-\alpha_v)^3}\ \frac{I_{min}}{1-\alpha_b{\alpha_h}^3} \tag{15}$$

$$N_y(\lambda)=EA(\lambda)\varepsilon_y=\frac{(1-\alpha_b)(1-\alpha_v\ \lambda)+\alpha_b(1-\alpha_H)}{(1-\alpha_v)(1-\alpha_b\alpha_h)}\ N_{y.min} \tag{16}$$

$$M_y(\lambda)=\frac{I(\lambda)\,\sigma_y}{h(\lambda)/2}=\frac{(1-\alpha_v\ \lambda)^3-\alpha_b(\alpha_H-\alpha_v\ \lambda)^3}{(1-\alpha_v)^3}\ \frac{I_{min}}{1-\alpha_b{\alpha_h}^3}\ E\varepsilon_y\ \frac{1-\alpha_v}{1-\alpha_v\ \lambda}\ \frac{2}{h} \tag{17}$$

The variables $N_y(\lambda)$ and $M_y(\lambda)$ may also be written according to the reduced forms (18) and (19):

$$n_y(\lambda)=\frac{N_y(\lambda)}{N_{y.min}}=\frac{(1-\alpha_b)(1-\alpha_v\ \lambda)+\alpha_b(1-\alpha_H)}{(1-\alpha_v)(1-\alpha_b\alpha_h)} \tag{18}$$

$$m_y(\lambda)=\frac{M_y(\lambda)}{M_{y.min}}=\frac{(1-\alpha_v\ \lambda)^3-\alpha_b(\alpha_H-\alpha_v\ \lambda)^3}{\left(1-\alpha_b{\alpha_h}^3\right)(1-\alpha_v)^2\ (1-\alpha_v\ \lambda)} \tag{19}$$

3. DISTRIBUTION OF THE ACTING CROSS-SECTION INTERNAL FORCES

As mentioned before, the application of this method is presented in the particular case of a tapered cantilever with a rectangular cross-section, subjected to concentrated axial and transversal loads at its free end (fig 4).

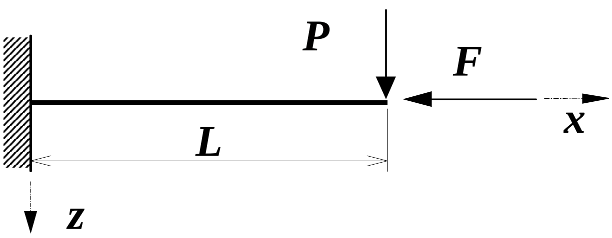

Fig. 4 – Dimensional variables

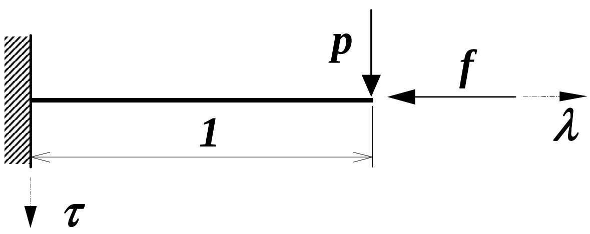

Fig. 5 – Reduced variables

The distribution of the acting axial force and bending moments may be written in the following non-dimensional forms:

$$n(\lambda)=\frac{N(\lambda)}{N_y(\lambda)}=\frac{F}{N_{y.min}}\ \frac{(1-\alpha_v)}{(1-\alpha_v\ \lambda)}=f\ \frac{(1-\alpha_v)}{(1-\alpha_v\ \lambda)} \tag{20}$$

$$m(\lambda)=\frac{M(\lambda)}{M_y(\lambda)}=\frac{P\,L\,(1-\lambda)}{M_{y.min}}\ \frac{(1-\alpha_v)^2}{(1-\alpha_v\ \lambda)^2}=p\ \frac{(1-\lambda)(1-\alpha_v)^2}{(1-\alpha_v\ \lambda)^2} \tag{21}$$

4. CRITICAL SECTION OF THE TAPERED BEAM-COLUMN

The critical section of the member is defined as the cross-section where the elastic limit state of the material is reached for the first time. This critical section may be found by searching the co-ordinate λ of the cross-section where the maximum stress reaches the yield strength of the material, σ_y. This condition may be written in the following reduced form:

$$\varsigma_{max}(\lambda) = 1 \tag{22}$$

where the variable $\varsigma_{max}(\lambda) = \sigma_{max}(\lambda)/\sigma_y$ represents the reduced value of the maximum stress $\sigma_{max}(\lambda)$ [1]. The value of $\varsigma_{max}(\lambda)$ may be determined by the addition of the absolute values of the reduced axial force $n(\lambda)$ and bending moment $m(\lambda)$, since the cross-sections are symmetrical about their strong axis of inertia [1, 3].

In the particular case of uniform cantilever beam, the critical section is always located at the beam fixed end (λ = 0), where the bending moment reaches its maximum value. In the case of a tapered beam, the resisting bending moment $M_y(\lambda)$ changes along the beam length and the critical section may correspond to another cross-section, different from λ = 0.

A more detailed explanation for finding the beam critical section may be found in reference [2].

5. CASE STUDY: TAPERED BEAM SUBMITTED TO CONCENTRATED LOADS

5.1 Brief description of the load combinations

In this paper, three different scenarios are studied, associated to a tapered beam-column with rectangular cross-sections submitted to the loading case described previously (fig. 3 and 4).

In the first case, the axial force F remains constant as the transversal force P is increased up to the elastic limit state of the tapered beam-column, $\varsigma_{max}(\lambda) = 1$. In the second case, the ratio between the values of F and P is kept constant and their values are increased up to $\varsigma_{max}(\lambda) = 1$. In the third case, the transversal force P is chosen constant and the axial force F increases up to the elastic limit state $\varsigma_{max}(\lambda) = 1$.

5.2 Constant axial force and increasing transversal force

The following expressions apply to tapered beam-columns with rectangular cross-sections submitted to a constant axial force and an increasing transversal force. As mentioned before, the elastic limit state of a cross-section λ is reached when the condition (22) is satisfied. This condition may be written in the following form [1]:

$$n(\lambda) + m(\lambda) = f\,\frac{(1-\alpha_v)}{(1-\alpha_v\,\lambda)} + p\,\frac{(1-\lambda)\,(1-\alpha_v)^2}{(1-\alpha_v\,\lambda)^2} = 1 \tag{23}$$

Therefore, the reduced value p_y of the transversal force at the elastic limit state of the cross-section λ is equal to:

$$p_y(\lambda) = \frac{(1-\alpha_v\,\lambda)^2}{(1-\lambda)\,(1-\alpha_v)^2}\left(1 - f\,\frac{(1-\alpha_v)}{(1-\alpha_v\,\lambda)}\right) \tag{24}$$

The critical section λ_y of the beam–column is the one where the elastic limit state is reached for the first time. So, the value of $p_y(\lambda)$ given by eq. (24) is minimum when $\lambda = \lambda_y$, and the combination of f and $p_y(\lambda_y)$ corresponds to the elastic limit state of the tapered beam.

The minimum value of $p_y(\lambda)$ may be obtained by the condition:

$$\frac{\partial p_y(\lambda)}{\partial \lambda} = 0 \quad \Rightarrow \quad \frac{1}{(1-\lambda)^2}\left(\frac{(1-\alpha_v\ \lambda)(1+\alpha_v\ \lambda-2\alpha_v)}{(1-\alpha_v)^2} - f\right) = 0 \tag{25}$$

The solutions of this equation are:

$$\lambda_{y1} = 1 + \frac{(1-\alpha_v)}{\alpha_v}\sqrt{1-f} \quad (26) \qquad\qquad \lambda_{y2} = 1 - \frac{(1-\alpha_v)}{\alpha_v}\sqrt{1-f} \quad (27)$$

The first solution does not lead to λ_y values within the limits of the beam length ($0 \le \lambda_y \le 1$). The second solution leads to a λ_y value satisfying the condition $0 \le \lambda_y \le 1$ when:

$$\alpha_v \ge \frac{\sqrt{1-f}}{1+\sqrt{1-f}} \tag{28}$$

If eq. (2) is introduced inserted in eq. (28), it is possible to conclude that the critical section is located at the fixed end of the tapered cantilever beam ($\lambda_y = 0$) if the ratio H/h between the height of the largest and smallest cross-sections is smaller than the limit indicated in eq. (29). Otherwise, the critical section is given by eq. (27):

$$\begin{cases} H/h \le 1+\sqrt{1-f} & \Rightarrow \quad \lambda_y = 0 \\ H/h > 1+\sqrt{1-f} & \Rightarrow \quad \lambda_y = 1 - \dfrac{(1-\alpha_v)}{\alpha_v}\sqrt{1-f} \end{cases} \tag{29}$$

The graphical representation of the position of the critical section λ_y, depending on the reduced value f of the axial force and on the ratio between the height of the largest and smallest cross-sections $H/h = 1/(1-\alpha_v)$, is represented on fig. 5.

In the elastic domain, the maximum value of the transversal load, $p_y(\lambda_y)$, is obtained by replacing the value of λ in eq. (24) by the value of λ_y given by eq. (29).

The value of $p_y(\lambda_y)$ represents the increase of the maximum transversal load value that can be obtained with a tapered beam, when compared to a uniform beam with a constant cross-section equal to the smallest tapered beam cross-section.

However, the evaluation of this value must take into account the steel quantity required to produce the tapered shape of the beam. The volume of the above mentioned uniform beam is noted Vol_{min} as the one of the tapered beam is Vol_{tap}. This last is given by eq. (30) [2]:

$$Vol_{tap} = A_{med}\ L = \frac{bh+bH}{2}L = \frac{bhL}{2}\left(1+\frac{1}{1-\alpha_v}\right) = \frac{2-\alpha_v}{2\,(1-\alpha_v)}Vol_{min} \tag{30}$$

Therefore, the increase of resistance of the tapered beam per volume unit, p^*, may be obtained by means of the following expression:

$$p^* = \frac{p_y(\lambda_y)}{1-f}\frac{Vol_{min}}{Vol_{tap}} = \frac{2\,(1-\alpha_v)}{(2-\alpha_v)\,(1-f)}\ p_y(\lambda_y) \tag{31}$$

5.3 Axial force and transversal force increasing proportionally

The following expressions apply to tapered beam-columns with rectangular cross-sections submitted to increasing concentrated axial and transversal loads. The ratio $\alpha_f = f/p$, between the reduced values of these loads, remains constant up to the beam-column elastic limit. This loading state is equivalent to a single concentrated load, inclined to the axial load of the beam-column, with two components, *f* and *p*.

The elastic limit state of a cross-section λ is still defined by eq. (23). In this case, the reduced value p_y of the transversal force at the elastic limit state of the cross-section λ is:

$$p_y(\lambda)=\frac{(1-\alpha_v\,\lambda)^2}{(1-\lambda)(1-\alpha_v)^2+\alpha_f\,(1-\alpha_v\,\lambda)(1-\alpha_v)} \tag{32}$$

The minimum value of $p_y(\lambda)$ may be obtained by the condition:

$$\frac{\partial p_y(\lambda)}{\partial\lambda}=0 \;\Rightarrow\; -\frac{(1-\alpha_v\,\lambda)\left((1-\alpha_v\,\lambda)(1-\alpha_v)+\alpha_f\,\alpha_v(1-\lambda)-2(1-\alpha_v)^2\right)}{(1-\alpha_v)(1-\lambda)^2\left(\alpha_f+1-\alpha_v\right)^2}=0 \tag{33}$$

The solutions of this equation are:

$$\lambda_{y1}=\frac{1}{\alpha_v} \tag{34}$$

$$\lambda_{y2}=\frac{1}{\alpha_v}\left(1-\frac{2\,(1-\alpha_v)^2}{1-\alpha_v+\alpha_f\,\alpha_v}\right) \tag{35}$$

The first solution does not lead to λ_y values within the limits of the beam length ($0\le\lambda_y\le1$). The second solution leads to a λ_y value satisfying the condition $0\le\lambda_y\le1$ when:

$$\frac{3+\alpha_f}{4}-\sqrt{\left(\frac{3+\alpha_f}{4}\right)^2-\frac{1}{2}}\le\alpha_v\le\frac{1}{1+\alpha_f} \tag{36}$$

Therefore, the critical section is at the fixed end ($\lambda_y = 0$) or at the free end ($\lambda_y = 1$) of the tapered cantilever beam if the ratio α_v is smaller or greater than the limits indicated in eq. (36); otherwise, the critical section is given by eq. (35):

$$\begin{cases}\alpha_v\le\dfrac{3+\alpha_f}{4}-\sqrt{\left(\dfrac{3+\alpha_f}{4}\right)^2-\dfrac{1}{2}} \;\Rightarrow\; \lambda_y=0\\[2ex] \dfrac{3+\alpha_f}{4}-\sqrt{\left(\dfrac{3+\alpha_f}{4}\right)^2-\dfrac{1}{2}}<\alpha_v<\dfrac{1}{1+\alpha_f} \;\Rightarrow\; \lambda_y=\dfrac{1}{\alpha_v}\left(1-\dfrac{2\,(1-\alpha_v)^2}{1-\alpha_v+\alpha_f\,\alpha_v}\right)\\[2ex] \alpha_v\ge\dfrac{1}{1+\alpha_f} \;\Rightarrow\; \lambda_y=1\end{cases} \tag{37}$$

The graphical representation of the position of the critical section λ_y of the tapered beam-column, depending on the inclination of the concentrated force (defined by α_f) and on the ratio between the height of the largest and smallest cross-sections $H/h=1/(1-\alpha_v)$, is shown on fig. 6.

In the beam-column elastic domain, the maximum value of the transversal load, $p_y(\lambda_y)$, is obtained by replacing the value of λ in eq. (32) by λ_y given by eq. (37). The increase of resistance of the tapered beam per volume unit, p^*, may be obtained by means of the following relationship, where Vol_{tap} represents the volume of the tapered beam, given by eq. (30) [2]:

$$p^*=p_y(\lambda_y)(1-\alpha_f)\frac{Vol_{min}}{Vol_{tap}}=\frac{2\,(1-\alpha_v)(1-\alpha_f)}{(2-\alpha_v)}\,p_y(\lambda_y) \tag{38}$$

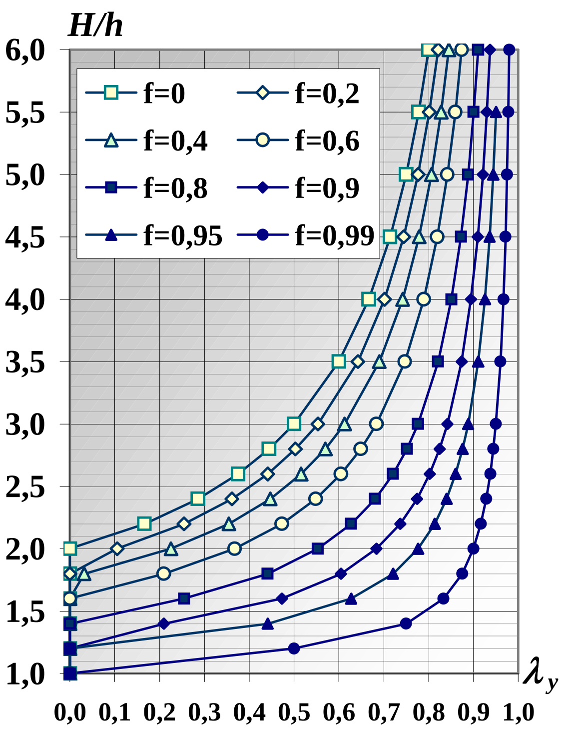

Fig. 5 – Location of the critical section in the case of constant axial load

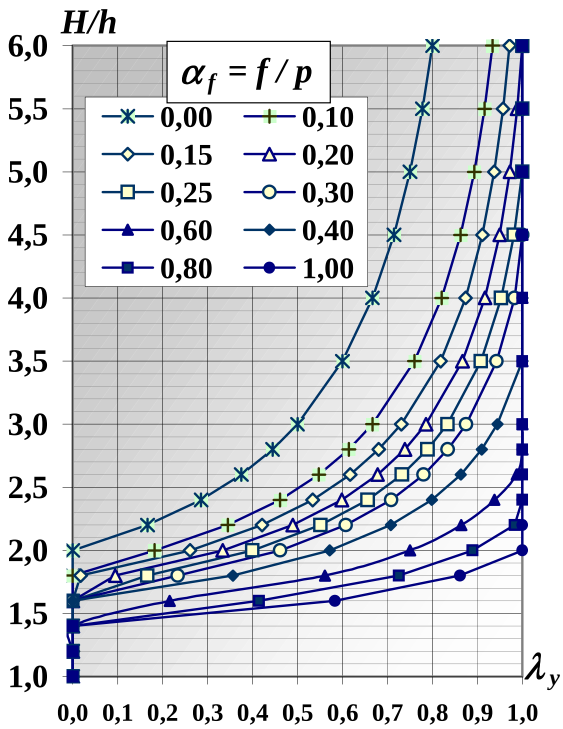

Fig. 6 – Location of the critical section in the case of proportional axial and transversal loads

5.4 Constant transversal force and increasing axial force

The following expressions apply to tapered beam-columns with rectangular cross-sections submitted to a constant transversal force and an increasing axial force. The elastic limit state of a cross-section λ is still defined by eq. (23). In this case, the reduced value f_y of the axial force at the elastic limit state of the cross-section λ is equal to:

$$f_y(\lambda) = \frac{1-\alpha_v\,\lambda}{1-\alpha_v}\left(1 - p\,(1-\lambda)\,\frac{(1-\alpha_v)^2}{(1-\alpha_v\,\lambda)^2}\right) \tag{39}$$

The minimum value of $f_y(\lambda)$ may be obtained by the condition:

$$\frac{\partial f_y(\lambda)}{\partial \lambda} = 0 \quad \Rightarrow \quad \frac{p\,(1-\alpha_v)^3 - \alpha_v\,(1-\alpha_v\,\lambda)^2}{(1-\alpha_v)(1-\alpha_v\,\lambda)^2} = 0 \tag{40}$$

whose solutions are:

$$\lambda_{y1} = \frac{1}{\alpha_v}\left(1 + (1-\alpha_v)\sqrt{\frac{p}{\alpha_v}(1-\alpha_v)}\right) \quad (41) \qquad \lambda_{y2} = \frac{1}{\alpha_v}\left(1 - (1-\alpha_v)\sqrt{\frac{p}{\alpha_v}(1-\alpha_v)}\right) \quad (42)$$

The first one does not lead to λ_y values within the limits of the beam length ($0 \le \lambda_y \le 1$). The second solution leads to a λ_y value satisfying the condition $\lambda_y \ge 0$ when eq. (43) is verified, and satisfying the condition $\lambda_y \le 1$ when eq. (44) is verified:

$$p \le \frac{\alpha_v}{(1-\alpha_v)^3} \quad (43) \qquad p \ge \frac{\alpha_v}{1-\alpha_v} \quad (44)$$

Therefore, if eq. (2) is included in (43) and (44), it is possible to conclude that:

$$\begin{cases} p \geq \left(\frac{H}{h}-1\right)\left(\frac{H}{h}\right)^2 & \Rightarrow \quad \lambda_y = 0 \\ \left(\frac{H}{h}-1\right) < p < \left(\frac{H}{h}-1\right)\left(\frac{H}{h}\right)^2 & \Rightarrow \quad \lambda_y = \frac{1}{\alpha_v}\left(1-(1-\alpha_v)\sqrt{\frac{p}{\alpha_v}(1-\alpha_v)}\right) \\ p \leq \left(\frac{H}{h}-1\right) & \Rightarrow \quad \lambda_y = 1 \end{cases} \tag{45}$$

In the elastic domain of the beam-column, the maximum value of the transversal load, $f_y(\lambda_y)$, is obtained by replacing the value of λ in eq. (39) by λ_y given by eq. (45).

The resistance increase of the tapered beam per volume unit, f^*, may be obtained by means of the following expression, where Vol_{tap} represents the volume of the tapered beam, given by eq. (30) [2]:

$$f^* = \frac{f_y(\lambda_y)}{1-p}\frac{Vol_{min}}{Vol_{tap}} = \frac{2\,(1-\alpha_v)}{(2-\alpha_v)(1-p)}\; f_y(\lambda_y) \tag{46}$$

6. EXAMPLES OF APPLICATION

In order to show the application of these analytical expressions, four examples are presented according to the previous case study.

6.1 The first one concerns a tapered cantilever beam-column whose length is L = 2000 mm. The cross-sections are rectangular, with a constant width b = 20 mm and a height presenting a linear variation from the free end, where h=180 mm, to the fixed end, where H = 360 mm. The yield strength of the material is σ_y = 240 MPa.

In this case, H/h = 2 and, according to eq. (2), α_v = 0,5. The properties of the smallest cross-section are A_{min} = 3600 mm^2, I_{min} = 972 cm^4, $N_{y.min}$ = 864 kN and $M_{y.min}$ = 25,92 kN.m.

The cross-sections are submitted to a constant axial load F = 432 kN. What is the maximum transversal concentrated load that can be applied at the member free-end, before the critical section starts yielding?

The reduced value of the axial load is f = 0,5 (20) and, according to eq. (29), the critical section is located at $\lambda_y = 1-\sqrt{0{,}5} = 0{,}2929$. So, the maximum reduced value of the transversal load (24) is p_y = 2,9142 and the dimensional value of this force is P = 37,768 kN (21).

Eq. (29) shows also that the critical section would be located at the fixed end of the tapered member (λ_y = 0) if $H \leq \left(1+\sqrt{0{,}5}\right)h$, or if $H \leq 307\ mm$. So, if the height of the largest cross-section is H = 300 mm (H/h = 5/3 and α_v = 0,4), for instance, then λ_y = 0 (29), p_y = 1,9444 (24) and the maximum transversal concentrated load at the free end is P = 25,200 kN (21).

6.2 The second example concerns the same tapered cantilever beam-column (H = 360 mm) submitted to a constant transversal concentrated load P = 25,92 kN. What maximum axial load can be applied at the member free-end, before its critical section starts yielding?

The reduced value of the transversal load is p = 2 (21) and according to eq. (45) the location of the critical section is $\lambda_y = \left(1-0{,}5\sqrt{2}\right) = 0{,}58579$. So, the maximum reduced value of the axial load (39) is f_y = 0,82843 and the dimensional value of this force is F = 715,761 kN (20).

6.3 The third example concerns the same tapered cantilever beam-column (with H = 360 mm) submitted to an inclined concentrated load with such a direction that the ratio between the reduced values of the axial and transversal components is $\alpha_f = f/p = 1/8$. What is the maximum value of this load, before its critical section starts yielding?

According to eq. (37), $\lambda_y = \left(1 - 2\times 0{,}5^2 / (1 - 0{,}5 + 0{,}5/8)\right)/0{,}5 = 0{,}22222$. So the maximum reduced value of the transversal load (39) is p_y = 3,16049 and $f_y = p_y/8$ = 0,39506. Therefore, the dimensional values of these components are F =3 41,333 kN (20) and P = 40,960 kN (21), and the maximum value of the inclined load is 343,782 kN.

6.4 Finally, the fourth example concerns a tapered cantilever beam-column with the same smallest cross-section, submitted to an inclined concentrated load with the same direction as in the previous example, $\alpha_f = f/p = 1/8$. What is the minimum value of the height of the largest section H, so that the critical section of the member starts to yield when the transversal component of the inclined load is P = 51,84 kN?

According to eq. (21), p_y = 4 and $f_y = p_y/8$ = 0,5; the value of the ratio α_v may be obtained using eq. (37) and (32), by means of a trial procedure. After a few (4 to 5) iterations, the value found for λ_y is 0,48484 (37) and α_v = 0,57852 (32); therefore, $H = h/(1-\alpha_v) = 427{,}1\ mm$.

7. CONCLUSIONS

This paper presents a method of deduction of analytical expressions for the elastic design of tapered beam-columns. A case study shows the application of this method to tapered beam-columns with rectangular cross-sections subjected to concentrated axial and transversal loads. Three loading cases are studied. In the first one, the axial force remains constant while the transversal force increases up to the elastic limit state of the member. In the second case, the ratio between the values of the two loads is kept constant while their resultant grows. In the third case, the transversal force remains constant while the axial force is increased.

The proposed relationships allow the way how the various parameters defining a tapered member act together to be understood. It is also possible, for instance, to define the best geometry of the member to resist a certain loading combination, or to determine the load intensities leading a beam-column with a given geometry to its elastic limit state. Some examples are presented to emphasise these possibilities.

This method may also be used for other loading cases, or for members with other kinds of cross-sections (as I- or H-cross-sections, for instance) or different boundary conditions. It just needs to be adapted to the new distributions of acting and resisting cross-section internal forces.

8. REFERENCES

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