SIMPLIFICAÇÃO DOS POTENCIAIS HARMÓNICOS RECORRENDO ÀS DISTRIBUIÇÕES

TEOREMA DA REPRESENTAÇÃO

$$\phi(\vec{x}) = -\frac{1}{4\pi}\int_S \frac{\partial}{\partial n(\vec{y})}\left(\frac{1}{|\vec{y}-\vec{x}|}\right)\phi(\vec{y})dS_{\vec{y}} + \frac{1}{4\pi}\int_S \frac{1}{|\vec{y}-\vec{x}|}\frac{\partial\phi(\vec{y})}{\partial n(\vec{y})}dS_{\vec{y}} \quad ; \quad \vec{x}\in V, \vec{y}\in S$$

POTÊNCIAIS DE CAMADA SIMPLES
(PROBLEMA DE NEUMANN)

a)Equação de Derivadas Parciais

$$\nabla^2_{\vec{x}}\phi + f(\vec{x}) = 0 \quad ; \quad f(\vec{x}) = \int_S \frac{\partial\phi(\vec{y})}{\partial n(\vec{y})}\delta(\vec{y}-\vec{x})dS_{\vec{y}} \quad ; \quad \vec{x}\in V; \vec{y}\in S$$

No Problema Interior a condição necessária para o equilíbrio é $\int_S \int_S \delta(\vec{y}-\vec{x})\phi(\vec{y})_{;i} n^i(\vec{y})\, dS_{\vec{y}} dS_{\vec{x}} = 0$. A solução fica determinada a menos de uma constante.

No Problema Exterior não existe condição necessária para o equilíbrio. A solução é única se satisfazer a condição de radiação.

b)Potencial de Camada Simples

$$\phi(\vec{x}) = \frac{1}{4\pi}\int_S \frac{1}{|\vec{y}-\vec{x}|}\varphi(\vec{y})dS_{\vec{y}} \quad ; \quad \vec{x}\in V; \vec{y}\in S$$

em que:

$$\int_S \delta(\vec{y}-\vec{x})\varphi(\vec{y})dS_{\vec{y}} + \frac{1}{4\pi}\int_S \frac{\partial}{\partial n(\vec{x})}\left(\frac{1}{|\vec{y}-\vec{x}|}\right)\varphi(\vec{y})dS_{\vec{y}} = f(\vec{x}) \quad ; \quad \vec{x}, \vec{y}\in S$$

Pelo Teorema da Representação [Eringen, A.C.; Suhubi, E.S.]:

$$\phi(\vec{x}) = \frac{1}{4\pi}\int_S \frac{1}{|\vec{y}-\vec{x}|}\frac{\partial\phi(\vec{y})}{\partial n(\vec{y})}dS_{\vec{y}} \quad ; \quad \vec{x}\in V, \vec{y}\in S$$

vem:

$$\varphi(\vec{y}) = \frac{\partial\phi(\vec{y})}{\partial n(\vec{y})} \quad ; \quad \vec{x} \in V, \vec{y} \in S$$

Por outro lado:

$$\nabla^2_{\vec{x}}\phi(\vec{x}) = -f(\vec{x})$$
$$= -\int_S \delta(\vec{y}-\vec{x})\frac{\partial\phi(\vec{y})}{\partial n(\vec{y})}dS_{\vec{y}} \quad \Rightarrow \quad \int_S \delta(\vec{y}-\vec{x})\varphi(\vec{y})dS_{\vec{y}} = f(\vec{x}) \quad ; \quad \vec{x} \in V, \vec{y} \in S$$

Assim:

$$\phi(\vec{x}) = \frac{1}{4\pi}\int_S \frac{1}{|\vec{y}-\vec{x}|}\varphi(\vec{y})dS_{\vec{y}} \quad ; \quad \vec{x} \in V; \vec{y} \in S$$

e o integral

$$\frac{1}{4\pi}\int_S \frac{\partial}{\partial n(\vec{x})}\frac{1}{|\vec{y}-\vec{x}|}\varphi(\vec{y})dS_{\vec{y}} \quad ; \quad \vec{x}; \vec{y} \in S$$

anula-se.

POTÊNCIAIS DE CAMADA DUPLA (PROBLEMA DE DIRICHLET)

a)Equação de Derivadas Parciais

$$\nabla^2_{\vec{x}}\phi + \frac{\partial f(\vec{x})}{\partial n(\vec{x})} = 0 \quad ; \quad f(\vec{x}) = \int_S \phi(\vec{y})\,\delta(\vec{y}-\vec{x})dS_{\vec{y}} \quad ;$$
$$\frac{\partial f(\vec{x})}{\partial n(\vec{x})} = \int_S \frac{\partial\phi(\vec{y})}{\partial n(\vec{x})}\delta(\vec{y}-\vec{x})dS_{\vec{y}} \quad ; \quad \vec{x} \in V; \vec{y} \in S$$

b)Potencial de Camada Dupla

$$\phi(\vec{x}) = \frac{1}{4\pi}\int_S \frac{\partial}{\partial n(\vec{y})}\left(\frac{1}{|\vec{y}-\vec{x}|}\right)\psi(\vec{y})dS_{\vec{y}} \quad ; \quad \vec{x} \in V; \vec{y} \in S$$

em que:

$$-\int_S \delta(\vec{y}-\vec{x})\psi(\vec{y})dS_{\vec{y}} + \frac{1}{4\pi}\int_S \frac{\partial}{\partial n(\vec{y})}\left(\frac{1}{|\vec{y}-\vec{x}|}\right)\psi(\vec{y})dS_{\vec{y}} = f(\vec{x}) \quad ; \quad \vec{x}, \vec{y} \in S$$

$\Rightarrow$

$$-\int_S \delta(\vec{y}-\vec{x})\frac{\partial\psi(\vec{y})}{\partial n(\vec{x})}dS_{\vec{y}} + \frac{1}{4\pi}\int_S \frac{\partial}{\partial n(\vec{x})}\frac{\partial}{\partial n(\vec{y})}\left(\frac{1}{|\vec{y}-\vec{x}|}\right)\psi(\vec{y})dS_{\vec{y}} = \frac{\partial f(\vec{x})}{\partial n(\vec{x})} \quad ; \quad \vec{x},\vec{y}\in S$$

Pelo Teorema da Representação [Eringen, A.C.; Suhubi, E.S.]:

$$\phi(\vec{x}) = -\frac{1}{4\pi}\int_S \frac{\partial}{\partial n(\vec{y})}\left(\frac{1}{|\vec{y}-\vec{x}|}\right)\phi(\vec{y})dS_{\vec{y}} \quad ; \quad \vec{x}\in V, \vec{y}\in S$$

vem:

$$\psi(\vec{y}) = -\phi(\vec{y}) \quad ; \quad \vec{x}\in V, \vec{y}\in S$$

Por outro lado:

$$\nabla^2_{\vec{x}}\phi(\vec{x}) = -\frac{\partial f(\vec{x})}{\partial n(\vec{x})}$$

$$= -\int_S \frac{\partial}{\partial n(\vec{x})}\delta(\vec{y}-\vec{x})\phi(\vec{y})\,dS_{\vec{y}}$$

$$\Rightarrow \quad \int_S \frac{\partial}{\partial n(\vec{x})}\delta(\vec{y}-\vec{x})\psi(\vec{y})dS_{\vec{y}} = -\frac{\partial f(\vec{x})}{\partial n(\vec{x})} \quad ; \quad \vec{x}\in V, \vec{y}\in S$$

$$\Rightarrow \frac{\partial\psi(\vec{x})}{\partial n(\vec{x})} = -\frac{\partial f(\vec{x})}{\partial n(\vec{x})}$$

e o integral

$$\frac{1}{4\pi}\int_S \frac{\partial}{\partial n(\vec{x})}\frac{\partial}{\partial n(\vec{y})}\left(\frac{1}{|\vec{y}-\vec{x}|}\right)\psi(\vec{y})dS_{\vec{y}} \; ; \vec{x},\vec{y}\in S$$

anula-se.

Pelo que $\phi(r,\theta,\varphi)=\dfrac{\Phi(r,\theta)}{2\pi}$. Simetria em θ implica:

$$\frac{\partial^2\Psi(r)}{(\partial r)^2}+\frac{2}{r}\frac{\partial\Psi}{\partial r}=\delta(r-1) \quad ; \quad \Psi(r)=\int_0^\pi \Phi(r,\theta)\,d\theta$$

Assim $\phi(r,\theta,\varphi)=\dfrac{\Psi(r)}{2\pi^2}$. Efectuando a transformada de Laplace em relação a r:

$$-\left(\frac{d}{ds}\right)\left(s^2\hat{\Psi}(s)\right)+2s\hat{\Psi}(s)=\frac{-j}{2\pi}\left(\frac{1!}{s^2}\right)*\left(s^2\hat{\Psi}(s)\right)+2s\hat{\Psi}(s)=e^{-s} \quad ; \quad L(r^n)=\left(\frac{n!}{s^{n+1}}\right)$$

Visto: $L(\delta(r-1))=e^{-s}$; $e^{-s}=1-s+\ldots+(-1)^n s^n/n!+$; $L^{-1}(e^{-s})=\delta(r)-\delta^{(1)}(r)+\ldots+(-1)^n\delta^{(n)}(r)/n!$

resolve-se em relação a $\Psi(r)=\sum_{n=0}\psi^n(r)$; $\left(\dfrac{\partial^2}{(\partial r)^2}+\dfrac{2}{r}\dfrac{\partial}{\partial r}\right)\left(\psi^n(r)\right)=\dfrac{(-1)^n}{n!}\delta^{(n)}(r)$ e portanto obtém-se $\phi(r,\theta,\varphi)$.

TRANSLAÇÕES

Seja:

$$\phi^{;i}_{;i}(x^m)=\delta(x^m-x^m(\bar{y}^p)) \quad ; \quad z^m=x^m-x^m(\bar{y}^p)$$

Então:

$$\phi^{;i}_{;i}(z^m)=\delta(z^m) \quad \Rightarrow \quad x^m=z^m+x^m(\bar{y}^p)$$

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Anexo

Nota:

$$\delta(J\vec{x})=\delta(\phi\lambda\bar{\phi}^T\vec{x})=\bar{\delta}(\overline{\phi\lambda}\phi^T\vec{x})$$

$$\delta(J^T\vec{x})=\delta(\bar{\phi}\lambda\phi^T\vec{x})=\bar{\delta}(\phi\overline{\lambda\phi}^T\vec{x}) \quad ; \quad \vec{x}\in R^n$$

$$\left[\begin{array}{l} J\to(\phi,\lambda)\to(\bar{\phi},\bar{\lambda}) \\ J^T\to(\phi,\bar{\lambda})\to(\bar{\phi},\lambda)\end{array}\right. \quad ; \quad \delta=\bar{\delta}$$

visto que:

$$\bar{\delta}((A+jB)\vec{x})=\delta((A-jB)\vec{x}) \quad ; \quad |A|>0 \quad ; \quad |B|>0$$

$$\delta((A+jB)\vec{x})=\bar{\delta}((A-jB)\vec{x})$$